Integral Inequalities for Some Convex Functions

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Abstract

In this paper, we established some new integral inequalities for different kinds of convex functions by using some classical inequalities.

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1. Introduction

We recall following definitions.

The functions $f: [a, b] \to \mathbb{R}$, is said to be convex, if we have

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$. We can define starshaped functions on $[0, b]$ which satisfy the condition

$$f(tx) \leq tf(x)$$

for $t \in [0, 1]$. TOADER (1984) defined the concept of $m$–convexity as the following:

Definition 1. The function $f: [a, b] \to \mathbb{R}$ is said to be $m$–convex, where $m \in [0, 1]$, if for every $x, y \in [a, b]$ and $t \in [0, 1]$, we have:

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y).$$

Denote by $K_m(b)$ the set of the $m$–convex functions on $[0, b]$ for which $f(0) \leq 0$.

Some interesting and important inequalities for $m$–convex functions can be found in our references.

HUDZIK and MALIGRANDA (1994) considered among others the class of functions which are $s$–convex in the second sense.

Definition 2. A function $f: \mathbb{R}^+ \to \mathbb{R}$ where $\mathbb{R}^+ = [0, \infty)$, is said to be $s$–convex in the second sense if

$$f(ax + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in [0, \infty)$, $a, \beta \geq 0$ with $a + \beta = 1$ and for some fixed $s \in (0, 1]$. This class of $s$–convex functions in the second sense is usually denoted by $K^s_2$.

$s$–convexity introduced by BRECKNER (1978) as a generalization of convex functions. Also, BRECKNER (1993) proved the fact that the set valued map is $s$–convex only if the associated support function is $s$–convex function.

DRAGOMIR and FITZPATRICK (1999) proved the following Hadamard type integral inequality:

Theorem 1. Suppose that $f: [0, \infty) \to [0, \infty)$ is an $s$–convex function in the second sense, where $s \in (0, 1]$ and let $a, b \in [0, \infty)$, $a < b$. If $f \in L_1[0,1]$, then the following inequalities hold:

$$2^{s-1} f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{s + 1}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.1). The above inequalities are sharp.

Several properties of $s$–convexity in the first sense are discussed in the paper that is written by HUDZIK and MALIGRANDA (1994). Obviously, $s$–convexity means just convexity when $s = 1$. Some new Hermite Hadamard type inequalities based on concavity and $s$–convexity established by KIRMACI et al. (2007). For related results see the papers DRAGOMIR and FITZPATRICK (1999) and KIRMACI et al (2007).

DRAGOMIR (2002) proved the following theorem.
Theorem 2. Let \( f: [0, \infty) \rightarrow \mathbb{R} \) be an \( m - \) convex function with \( m \in (0,1] \) and \( 0 \leq a \leq b \). If \( f \in L_1[0,1] \), then one has the inequalities

\[
(1.2) \quad f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b \frac{f(x) + mf\left(\frac{x}{m}\right)}{2} \, dx
\]

\[
\leq \frac{m + 1}{4} \left[ f(a) + f(b) + m \left( f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right) \right) \right]
\]

MIHEŞAN (1993) gave definition of \((a, m) - \) convexity as following:

Definition 3. The function \( f: [0, b) \rightarrow \mathbb{R} \), \( b > 0 \) is said to be \((a, m) - \) convex, where \( (a, m) \in [0,1]^2 \), if we have

\[
f(tx + m(1-t)y) \leq t^m f(x) + m(1-t)^m f(y)
\]

for all \( x, y \in [0, b] \) and \( t \in [0,1] \). Denote by \( K_a^m(b) \) the class of all \((a, m) - \) convex functions on \([0, b)\) for which \( f(0) \leq 0 \). If we choose \((a, m) = (1,1)\), we have ordinary convex functions on \([0, b)\). For the recent results based on the above definition see the papers BAKULA et al. (2006), BAKULA et al. (2008), ÖZDEMİR et al. (2010), SARIKAYA et al. (2011), SET et al. (2009), ÖZDEMİR et al. (2011).

Definition 4. (See PEČARIĆ et al. (1992)) A function \( f: I \rightarrow [0, \infty) \) is said to be \( \log - \) convex or multiplicatively convex if \( \log f \) is convex or equivalently if for all \( x, y \in I \) and \( t \in [0,1] \) one has the inequality:

\[
(1.3) \quad f(tx + (1-t)y) \leq [f(x)]^t[f(y)]^{1-t}
\]

We note that a \( \log - \) convex function is convex, but the converse may not necessarily be true.

Theorem 3. (ÖZDEMİR et al. (2010)) Let \( f, g: [a, b] \rightarrow \mathbb{R} \) be real valued non-negative convex functions and \( F(x,y)(t), G(x,y)(t): [0,1] \rightarrow \mathbb{R}^+ \) are defined as the followings:

\[
F(x,y)(t) = \frac{1}{2} \left[ f(tx + (1-t)y) + f((1-t)x + ty) \right]
\]

\[
G(x,y)(t) = \frac{1}{2} \left[ g(tx + (1-t)y) + g((1-t)x + ty) \right]
\]

for all \( t \in [0,1] \), we have

\[
(1.4) \quad \frac{1}{b - a} \int_a^b F\left( x, \frac{a + b}{2} \right) \, dx \leq \frac{1}{4} \int_a^b f(x)g(x) \, dx + \frac{3}{16} [M(a,b) + N(a,b)]
\]

And

\[
(1.5) \quad \frac{2}{(b - a)^2} \int_a^b \frac{F(x,y)(t)G(x,y)(t) \, dx \, dy}{f(x)g(x) \, dx + \frac{1}{4} [M(a,b) + N(a,b)]}
\]

where

\[
M(a,b) = f(a)g(a) + f(b)g(b)
\]

\[
N(a,b) = f(a)g(b) + f(b)g(a).
\]

The main purpose of this paper is to prove some new inequalities as above, but now for \( m- \) convex and \( s- \) convex functions by modified the mappings \( F(x,y)(t) \) and \( G(x,y)(t) \).

2. Main Results

Theorem 4. Let \( f, g: [0, \infty) \rightarrow \mathbb{R}^+ \) be \( m- \) convex functions with \( m \in (0,1] \), \( 0 \leq a < b \) and \( f, g, fg \in L_1[a,b] \). \( F(x,y)(t), G(x,y)(t): [0,1] \rightarrow \mathbb{R}^+ \) are defined as the followings:

\[
F(x,y)(t) = \frac{1}{2} \left[ ftx + m(1-t)y \right] + f((1-t)x + mt)
\]

\[
G(x,y)(t) = \frac{1}{2} \left[ gtx + m(1-t)y \right] + g((1-t)x + mt)
\]

for all \( t \in [0,1] \), we have

\[
(2.1) \quad \int_a^b f\left( x, \frac{a + b}{2} \right) \, dx + \frac{m^2}{4} (b - a) \mu_1 \mu_2 + \frac{m}{4} \mu_3 \int_a^b \frac{f(x)g(x)}{dx + \mu_2} \, dx \]

where

\[
\mu_1 = \frac{m + 1}{4} \left( \frac{g(a) + g(b)}{2} + \frac{g\left( \frac{a}{m} \right) + g\left( \frac{b}{m} \right)}{2} \right)
\]
\[ \mu_2 = \frac{m + 1}{4} \left( \frac{f(a) + f(b)}{2} + m \frac{f \left( \frac{a}{m} \right) + f \left( \frac{b}{m} \right)}{2} \right) \]

and

(2.2)

\[ \frac{2}{(b - a)^2} \int_a^b \int_a^b F(x, y)(t) G(x, y)(t) \, dx \, dy \]
\[ \leq \frac{m^2 + 1}{2} \int_a^b f(x) g(x) \, dx + \frac{m}{b - a} \int_a^b f(x) \, dx \int_a^b g(y) \, dy. \]

**Proof.** Since \( f \) and \( g \) are \( m \)-convex functions, we can write

\[ F(x, y)(t) \]
\[ \leq \frac{1}{2} \left[ tf(x) + m(1 - t)f(y) + (1 - t)f(x) + mtf(y) \right] \]
\[ = \frac{1}{2} \left[ f(x) + mf(y) \right], \]

(2.3)

\[ G \left( x, \frac{a + b}{2} \right)(t) \leq \frac{1}{2} \left[ g(x) + mg \left( \frac{a + b}{2} \right) \right] \]

and analogously, if we set \( x = x \) and \( y = \frac{a + b}{2} \), we can write

\[ G(x, y)(t) \]
\[ \leq \frac{1}{2} \left[ tg(x) + m(1 - t)g(y) + (1 - t)g(x) + mtg(y) \right] \]
\[ = \frac{1}{2} \left[ g(x) + mg(y) \right], \]

(2.4)

\[ G \left( x, \frac{a + b}{2} \right)(t) \leq \frac{1}{2} \left[ g(x) + mg \left( \frac{a + b}{2} \right) \right] \]

By multiplying the inequalities (2.3) and (2.4), we get

(2.5)

\[ F \left( x, \frac{a + b}{2} \right)(t) G \left( x, \frac{a + b}{2} \right)(t) \]
\[ \leq \frac{1}{4} \left[ f(x) + mf \left( \frac{a + b}{2} \right) \right] \left[ g(x) + mg \left( \frac{a + b}{2} \right) \right] \]
\[ = \left[ f(x)g(x) + mf \left( \frac{a + b}{2} \right)g(x) + mg \left( \frac{a + b}{2} \right)f(x) + m^2f \left( \frac{a + b}{2} \right)g \left( \frac{a + b}{2} \right) \right] \]

Integrating the above inequality with respect to \( x \) on \([a, b] \), we obtain the following inequality:

(2.6)

\[ \int_a^b F \left( x, \frac{a + b}{2} \right)(t) G \left( x, \frac{a + b}{2} \right)(t) \, dx \]
\[ \leq \frac{1}{4} \left\{ \int_a^b f(x)g(x) \, dx + \int_a^b mf \left( \frac{a + b}{2} \right)g(x) \, dx \right\} \]
\[ + \int_a^b mg \left( \frac{a + b}{2} \right)f(x) \, dx + m^2 \int_a^b f \left( \frac{a + b}{2} \right)g \left( \frac{a + b}{2} \right) \, dx \]

Using the inequalities in (1.2) and by rewriting the (2.6), the proof is completed.

**Remark 1.** If we choose \( m = 1 \), inequalities (2.1) and (2.2) reduces to (1.4) and (1.5) respectively.

**Theorem 5.** Let \( f, g : [0, \infty) \to \mathbb{R} \) be \( s \)-convex functions in the second sense and \( F(x, y)(t), G(x, y)(t) : [0, 1] \to \mathbb{R} \) are defined as the following:

\[ F(x, y)(t) = \frac{1}{2} \left[ f(t^s x + (1 - t)^s y) + f((1 - t)^s x + t^s y) \right] \]
\[ G(x, y)(t) = \frac{1}{2} \left[ g(t^s x + (1 - t)^s y) + f((1 - t)^s x + t^s y) \right] \]

If \( f, g, fg \in L_1[a, b] \), for all \( t \in [0, 1] \), we have

\[ \int_0^1 \left[ F(a, b)(t) + G(a, b)(t) \right] \, dt \]
\[ \leq \frac{f(a) + f(b) + g(a) + g(b)}{s + 1} \]

and

\[ \int_0^1 F(a, b)(t) \, G(a, b)(t) \, dt \]
\[ \leq \left[ M(a, b) + N(a, b) \right] \frac{1}{2(2s + 1)} + \frac{1}{2} \beta(s + 1, s + 1) \]

where

\[ M(a, b) = f(a)g(a) + f(b)g(b) \]
\[ N(a, b) = f(a)g(b) + f(b)g(a) \]

and the Euler Beta function is defined by

\[ \beta(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} \, dt, \quad x, y > 0. \]

**Proof.** Since \( f \) and \( g \) are \( s \)-convex functions in the second sense, we can write

(2.7)

\[ F(x, y)(t) \leq \frac{e^{f(x)(1-t)^s f(y) + (1-t)^s f(x) + t^s f(y)}}{2} \]
If we set \( x = a, y = b \) in the above inequalities and by addition, then by integrating with respect to \( t \) over \([0,1]\), we get:

\[
\int_0^1 [F(a, b)(t) G(a, b)(t)] dt \\
\leq \left[ \frac{f(a) + f(b) + g(a) + g(b)}{s + 1} \right] \\
= \frac{f(a) + f(b) + g(a) + g(b)}{s + 1}
\]

This completes the proof of the first inequality.

For the proof of the second inequality, if we multiply the inequalities (2.7) and (2.8) for \( \alpha = a, \beta = b \) and by integrating with respect to \( t \) over \([0,1]\), we have

\[
\int_0^1 [F(a, b)(t) G(a, b)(t)] dt \\
= [M(a, b) + N(a, b)] \left[ \frac{1}{2(2s + 1)} + \frac{1}{2} \beta(s + 1, s + 1) \right]
\]

The proof is completed.

Theorem 6. Let \( f, g: [0, \infty) \to \mathbb{R}_+ \) be \((\alpha, m)\)-convex functions with \((\alpha, m) \in (0,1)^2 \), \( 0 \leq a < b \) and \( f, g, f g \in L_1[a, b] \). \( F(x, y)(t), G(x, y)(t); [0,1] \to \mathbb{R}_+ \) are defined as the followings:

\[
F(x, y)(t) = \frac{1}{2} [f(tx + m(1-t)y) + f(m(1-t)x + ty)]
\]

\[
G(x, y)(t) = \frac{1}{2} [g(tx + m(1-t)y) + g(m(1-t)x + ty)]
\]

for all \( t \in [0,1] \), we have

\[
\int_0^1 [F(a, b)(t) + G(a, b)(t)] dt \\
\leq \frac{1}{2} \left[ 1 + ma \right] \left[ f(a) + f(b) + g(a) + g(b) \right]
\]

and

\[
\int_0^1 [F(a, b)(t) G(a, b)(t)] dt \\
\leq \frac{1}{4} [M(a, b) + N(a, b)] \left[ \frac{a(2m + m^2)}{a + 1} + \frac{m^2 + 1}{2a + 1} \right]
\]

where \( M(a, b) \) and \( N(a, b) \) as in Theorem 5.

Proof. Since \( f \) and \( g \) are \((\alpha, m)\)-convex functions, we can write

\[
F(x, y)(t) \leq \frac{t^a f(x) + (1-t)^a f(y) + (1-t)^a f(x) + t^a f(y)}{2}
\]

If we set \( x = a \) and \( y = b \), we get

\[
F(a, b)(t) \leq \frac{1}{2} [(f(a) + f(b))(t^a + m(1-t^a))]
\]

and analogously, we have

\[
G(a, b)(t) \leq \frac{1}{2} [(g(a) + g(b))(t^a + m(1-t^a))]
\]

By adding the inequalities (2.11) and (2.12), we get

(2.13)

\[
F(a, b)(t) + G(a, b)(t) \\
\leq \frac{1}{2} [(f(a) + f(b) + g(a) + g(b))(t^a + m(1-t^a))]
\]

Integrating the above inequality with respect to \( t \) on \([0,1]\), we obtain the inequality (2.9). For the proof of the inequality (2.10), by multiplying the inequalities (2.11) and (2.12), we have

\[
F(a, b)(t) G(a, b)(t) \\
\leq \frac{1}{4} [M(a, b) + N(a, b)] [t^{2a} + 2m t^a (1-t^a) + m^2 (1-t^a)^2]
\]

By integrating the above inequality with respect to \( t \) over \([0,1]\), we get the inequality (2.10).

Theorem 7. Let \( f, g: [0, \infty) \to \mathbb{R}_+ \) be logarithmically convex functions on \([0, \infty)\) and \( f, g, f g \in L_1[a, b] \). \( F(x, y)(t), G(x, y)(t); [0,1] \to \mathbb{R}_+ \) are defined as in Theorem 3, then the following inequalities hold:

(2.14)

\[
\int_0^1 [F(a, b)(t) + G(a, b)(t)] dt \\
\leq \frac{1}{2} \left[ L(f(a) g(a), f(b) g(b)) + L(f(a) g(b), f(b) g(a)) \right]
\]

for all \( t \in [0,1] \), where

\[
L(f(a) g(a), f(b) g(b)) = \frac{f(a) g(a) - f(b) g(b)}{ln f(a) g(a) - ln f(b) g(b)}
\]

and
\[
L(f(a)g(b), f(b)g(a)) = \frac{f(a)g(b) - f(b)g(a)}{\ln f(b) - \ln f(a)g(b)}.
\]

**Proof.** Since \(f, g\) are \(log -\) convex functions on \([a, b] \subseteq [0, \infty)\), we can write
\[
F(x, y)(t) \leq \frac{1}{2}[f^t(x) + f^{(1-t)}(y) + f^{(1-t)}(x) + f^t(y)]
\]
and
\[
G(x, y)(t) \leq \frac{1}{2}[g^t(x) + g^{(1-t)}(y) + g^{(1-t)}(x) + g^t(y)].
\]
If we set \(x = a\) and \(y = b\), we have
\[
F(x, y)(t) \leq \frac{1}{2}[f^t(a) + f^{(1-t)}(b) + f^{(1-t)}(a) + f^t(b)]
\]
and
\[
G(x, y)(t) \leq \frac{1}{2}[g^t(a) + g^{(1-t)}(b) + g^{(1-t)}(a) + g^t(b)].
\]
By multiplying the inequalities (2.15) and (2.16), we get
\[
F(a, b)(t)G(a, b)(t) \leq \frac{1}{4}[f^t(a) + f^{(1-t)}(b) + f^{(1-t)}(a) + f^t(b)]
\]
\[
\times [g^t(a) + g^{(1-t)}(b) + g^{(1-t)}(a) + g^t(b)].
\]
By integrating the above inequality with respect to \(t\) on \([0, 1]\), we obtain the inequality (2.14).

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