ON A SECOND ORDER RATIONAL DIFFERENCE EQUATION

Nouressadat Touafek∗

Received 06:07:2011 : Accepted 26:12:2011

Abstract

In this paper, we investigate the stability of the following difference equation

\[ x_{n+1} = \frac{ax_n^4 + bx_n^3x_{n-1} + cx_n^2x_{n-1}^2 + dx_nx_{n-1}^3 + ex_n^4}{Ax_n^4 + Bx_n^3x_{n-1} + Cx_n^2x_{n-1}^2 + Dx_nx_{n-1}^3 + Ex_n^4}, \]

where the parameters \( a, b, c, d, e, A, B, C, D, E \) are positive real numbers and the initial values \( x_0, x_{-1} \) are arbitrary positive numbers.

Keywords: Difference equations, Global stability.


1. Introduction and preliminaries

Consider the following second-order difference equation

\[ x_{n+1} = \frac{ax_n^4 + bx_n^3x_{n-1} + cx_n^2x_{n-1}^2 + dx_nx_{n-1}^3 + ex_n^4}{Ax_n^4 + Bx_n^3x_{n-1} + Cx_n^2x_{n-1}^2 + Dx_nx_{n-1}^3 + Ex_n^4}, \ n = 0, 1, \ldots,

where the initial conditions \( x_0, x_{-1} \in (0, \infty) \) and the parameters \( a, b, c, d, e, A, B, C, D, E \in (0, \infty) \). In this paper we study the global stability of the unique positive equilibrium point, and the boundedness and the convergence of the solutions of Equation (1.1).

Nonlinear difference equations appear naturally, for example, from certain models in ecology, economy, automatic control theory, and they are of great importance in applications where the \((n+1)^{\text{st}}\) state of the model depends on the previous \(k\) states.

Recently, there has been a lot of attention given to studying the global behavior of nonlinear difference equations by many authors, See for example [1-3,5-19] and the references cited therein.

Now, we review some definitions (see for example [11-12]), which will be useful in the sequel.

∗Department of Mathematics, Jijel University, Algeria.
E-mail: nstouafek@yahoo.fr
Let \( I \) be an interval of real numbers and let \( F : I \times I \rightarrow I \) be a continuously differentiable function. Consider the difference equation
\[
x_{n+1} = F(x_n, x_{n-1})
\]
with initial values \( x_{-1}, x_0 \in I \).

1.1. Definition. A point \( \overline{x} \in I \) is called an equilibrium point of (1.2) if \( \overline{x} = F(\overline{x}, \overline{x}) \).

1.2. Definition. Let \( \overline{x} \) be an equilibrium point of (1.2).
- The equilibrium \( \overline{x} \) is called locally stable if for every \( \epsilon > 0 \), there exist \( \delta > 0 \) such that for all \( x_{-1}, x_0 \in I \) with \( |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \delta \), we have \( |x_n - \overline{x}| < \epsilon \), for all \( n \geq -1 \).
- The equilibrium \( \overline{x} \) is called locally asymptotically stable if it is locally stable, and if there exists \( \gamma > 0 \) such that if \( x_{-1}, x_0 \in I \) and \( |x_{-1} - \overline{x}| + |x_0 - \overline{x}| < \gamma \) then
  \[
  \lim_{n \to +\infty} x_n = \overline{x}.
  \]
- The equilibrium \( \overline{x} \) is called a global attractor if for all \( x_{-1}, x_0 \in I \), we have
  \[
  \lim_{n \to +\infty} x_n = \overline{x}.
  \]
- The equilibrium \( \overline{x} \) is called global asymptotically stable if it is locally stable and a global attractor.
- The equilibrium \( \overline{x} \) is called unstable if it is not stable.
- Let \( p = \frac{\partial F}{\partial x}(\overline{x}, \overline{x}) \) and \( q = \frac{\partial F}{\partial y}(\overline{x}, \overline{x}) \). Then the equation
  \[
  y_{n+1} = py_n + qy_{n-1}, \quad n = 0, 1, \ldots
  \]
  is called the linearized equation of (1.2) about the equilibrium point \( \overline{x} \).

The next result, which was given by Clark [4], provides a sufficient condition for the locally asymptotically stability of (1.2).

1.3. Theorem. Consider the difference equation (1.3). Then, \( |p| + |q| < 1 \) is a sufficient condition for the locally asymptotically stability of (1.2). \( \square \)

1.4. Definition. The difference equation (1.2) is said to be permanent if there exist numbers \( \alpha, \beta \) with \( 0 < \alpha \leq \beta < \infty \) such that for any initial values \( x_{-1}, x_0 \in I \) there exists a positive integer \( N \) which depends on the initial conditions such that \( \alpha \leq x_n \leq \beta \) for all \( n \geq N \).

2. Main results

Let us define the following real numbers: \( r_1 = aB - bA, r_2 = aC - cA, r_3 = aD - dA, r_4 = aE - eA, r_5 = bE - eB, r_6 = cB - bC, r_7 = cE - eC, r_8 = dB - bD, r_9 = dC - cD, r_{10} = dE - eD \). 

2.1. Remark. Equation (1.1) has a unique positive equilibrium point which is given by
\[
\overline{x} = \frac{a + b + c + d + e}{A + B + C + D + E}.
\]

Let \( f : (0, +\infty)^2 \rightarrow (0, +\infty) \) be the function defined by
\[
f(x, y) = \frac{ax^4 + bxy^3 + cxy^2 + dx^3y + ey^4}{Ax^4 + Bxy^3 + Cxy^2 + Dxy^3y + Ey^4}.
\]
2.2. Lemma.

Assume that

(a) \( \frac{1}{x} \geq \text{max}(\frac{d}{x}, \frac{e}{y}) \),
(b) \( \frac{1}{x} \leq \text{min}(\frac{d}{x}, \frac{e}{y}) \),
(c) \( 3r_1 + r_9 \geq 0 \),
(d) \( 2r_4 + r_8 \geq 0 \),
(e) \( r_6 + 3r_{10} \geq 0 \).

Then \( f \) is increasing in \( x \) for each \( y \) and it is decreasing in \( y \) for each \( x \).

Assume that

(a) \( \frac{1}{x} \leq \text{min}(\frac{d}{x}, \frac{e}{y}) \),
(b) \( \frac{1}{x} \geq \text{max}(\frac{d}{x}, \frac{e}{y}) \),
(c) \( 3r_1 + r_9 \leq 0 \),
(d) \( 2r_4 + r_8 \leq 0 \),
(e) \( r_6 + 3r_{10} \leq 0 \).

Then \( f \) is decreasing in \( x \) for each \( y \) and it is increasing in \( y \) for each \( x \).

Proof. (1) We have, \( 3r_1 + r_9 \geq 0 \), \( 2r_4 + r_8 \geq 0 \) and \( r_6 + 3r_{10} \geq 0 \). Using the fact that, \( \frac{1}{x} \geq \text{max}(\frac{d}{x}, \frac{e}{y}) \) and \( \frac{1}{x} \leq \text{min}(\frac{d}{x}, \frac{e}{y}) \), we get: \( r_2, r_3, r_5, r_7 \geq 0 \). Now, the result follows from the formulas

\[
\frac{\partial f}{\partial x}(x, y) = \frac{r_3y^6 + 2r_2y^3x^5 + (r_9 + 3r_1)y^3x^4 + 4r_8y^4x^3 + (3r_{10} + r_6)y^4x^2 + (2r_7)y^6x + r_5y^7}{(Ax^4 + Bxy^3 + Cx^2y^2 + Dxy^4 + Ey^5)'^2},
\]

\[
\frac{\partial f}{\partial y}(x, y) = \frac{-r_3x^7 - 2r_2y^6 - (r_9 + 3r_1)y^3x^5 - 4r_8y^4x^3 - (3r_{10} + r_6)y^4x^2 - 2r_7y^5x^2 - r_5y^6x}{(Ax^4 + Bxy^3 + Cx^2y^2 + Dxy^4 + Ey^5)'^2}.
\]

(2) The proof of (2) is similar and will be omitted.

The locally stability of the positive equilibrium point \( \overline{x} = \frac{a + b + c + d + e}{A + B + C + D + E} \) of (1.1) is described in the following theorem.

2.3. Theorem. Assume that

\[
2 \frac{|3r_1 + 2r_2 + r_3 + 2r_4 + r_5 + r_6 + 2r_7 + 2r_8 + r_9 + 3r_{10}|}{(a + b + c + d + e)(A + B + C + D + E)} < 1.
\]

Then the positive equilibrium point \( \overline{x} = \frac{a + b + c + d + e}{A + B + C + D + E} \) of (1.1) is locally asymptotically stable.

Proof. The linearized equation of (1.1) about \( \overline{x} = \frac{a + b + c + d + e}{A + B + C + D + E} \) is

\[
x_{n+1} = px_n + qx_{n-1}
\]

where

\[
p = \frac{3r_1 + 2r_2 + r_3 + 2r_4 + r_5 + r_6 + 2r_7 + 2r_8 + r_9 + 3r_{10}}{(a + b + c + d + e)(A + B + C + D + E)}
\]

and

\[
q = -\frac{3r_1 + 2r_2 + r_3 + 2r_4 + r_5 + r_6 + 2r_7 + 2r_8 + r_9 + 3r_{10}}{(a + b + c + d + e)(A + B + C + D + E)}.
\]

By using Theorem 1.3, we get that \( \overline{x} \) is locally asymptotically stable if

\[
2 \frac{|3r_1 + 2r_2 + r_3 + 2r_4 + r_5 + r_6 + 2r_7 + 2r_8 + r_9 + 3r_{10}|}{(a + b + c + d + e)(A + B + C + D + E)} < 1.
\]
The next theorem is devoted to the permanence of the difference equation (1.1).

2.4. Theorem. Let \( \{x_n\}_{n=1}^{+\infty} \) be a positive solution of equation (1.1).

(1) Assume that
(a) \( \frac{c}{A} \geq \max\left(\frac{d}{A}, \frac{e}{A}, \frac{f}{A}\right) \),
(b) \( \frac{e}{A} \leq \min\left(\frac{d}{A}, \frac{e}{A}, \frac{f}{A}\right) \).

Then,
\[
\frac{c}{A} \leq x_n \leq \frac{a}{A}
\]
for all \( n \geq 1 \).

(2) Assume that
(a) \( \frac{d}{A} \leq \min\left(\frac{d}{A}, \frac{e}{A}, \frac{f}{A}\right) \),
(b) \( \frac{d}{A} \geq \max\left(\frac{d}{A}, \frac{e}{A}, \frac{f}{A}\right) \).

Then,
\[
\frac{a}{A} \leq x_n \leq \frac{e}{A}
\]
for all \( n \geq 1 \).

Proof. (1) We have
\[
x_{n+1} = \frac{a}{A} = \frac{-r_3 x_{n-1}^3 + r_2 x_{n-1}^2 - r_1 x_{n-1} - r_4 x_{n-1}}{A(x_n^4 + B x_n x_{n-1} + C x_n^2 + D x_{n-1} + E x_{n-1})}
\]
\[
x_{n+1} = \frac{c}{E} = \frac{r_4 x_n^4 + r_1 x_{n-1}^3 + r_2 x_{n-1}^2 + r_3 x_n^3 + r_5 x_{n-1}}{E(x_n^4 + B x_n x_{n-1} + C x_n^2 + D x_{n-1} + E x_{n-1})}
\]
Now, it suffices to get \( r_1, r_2, r_3, r_4, r_5, r_7, r_{10} \geq 0 \), which result from \( \frac{d}{A} \geq \max\left(\frac{d}{A}, \frac{e}{A}, \frac{f}{A}\right) \) and \( \frac{d}{A} \leq \min\left(\frac{d}{A}, \frac{e}{A}, \frac{f}{A}\right) \).

(2) Similarly we can easily prove (2).

Here we study the global asymptotic stability of equation (1.1).

2.5. Theorem. Let
\[
p_1 = (A + D)e + Ab - Ea,
p_2 = (A + C + D)e - Ed + Ac + (A + D)b - (B + E)a,
p_3 = (A + B + C + D)e + (A - B - E)d + (A + D - E)c + (A + C + D)b - (B + C + E)a,
p_4 = (A + B + C + D + E)e + (A - B - C + D - E)d + (A - B + C + D - E)c + (A + B + C + D - E)b + (A - B - C - D - E)a.
\]

Assume that
(1) \( \frac{d}{A} \geq \max\left(\frac{d}{A}, \frac{e}{A}, \frac{f}{A}\right) \),
(2) \( \frac{e}{A} \leq \min\left(\frac{d}{A}, \frac{e}{A}, \frac{f}{A}\right) \),
(3) \( 3r_1 + r_9 \geq 0 \),
(4) \( 2r_4 + r_8 \geq 0 \),
(5) \( r_6 + 3r_{10} \geq 0 \),
(6) \( 2\frac{[3r_1 + 2r_4 + r_9 + r_8 + r_3 + 2r_7 + 2r_8 + r_9 + 3r_{10}]}{(a+b+c+d+e)(A+B+C+D+E)} < 1 \),
(7) \( p_1, p_2, p_3, p_4 \geq 0 \).

Then the equilibrium point \( \mathbf{y} = \frac{a+b+c+d+e}{A+B+C+D+E} \) of (1.1) is globally asymptotically stable.

Proof. Let \( \{x_n\}_{n=1}^{+\infty} \) be a solution of equation (1.1). In view of theorem 2.3 we need only to prove that \( \mathbf{y} \) is a global attractor.

Let
\[
m = \lim_{n \to +\infty} \inf x_n
\]
So, it suffices to show that \( m = M \).

Let \( \varepsilon \in ]0, m[ \) then there exist \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \) we get

\[
m - \varepsilon \leq x_n \leq M + \varepsilon.
\]

Thus by using Lemma 2.2, Part (1); we get for all \( n \geq n_0 + 1 \)

\[
x_n + 1 \geq \frac{a(m - \varepsilon)^4 + b(m - \varepsilon)(M + \varepsilon)^3 + c(m - \varepsilon)^2(M + \varepsilon)^2 + d(m - \varepsilon)^3(M + \varepsilon) + e(M + \varepsilon)^4}{A(m - \varepsilon)^4 + B(m - \varepsilon)(M + \varepsilon)^3 + C(m - \varepsilon)^2(M + \varepsilon)^2 + D(m - \varepsilon)^3(M + \varepsilon) + E(M + \varepsilon)^4}
\]

Then we get the following inequalities

\[
m \geq \frac{a(m - \varepsilon)^4 + b(m - \varepsilon)(M + \varepsilon)^3 + c(m - \varepsilon)^2(M + \varepsilon)^2 + d(m - \varepsilon)^3(M + \varepsilon) + e(M + \varepsilon)^4}{A(m - \varepsilon)^4 + B(m - \varepsilon)(M + \varepsilon)^3 + C(m - \varepsilon)^2(M + \varepsilon)^2 + D(m - \varepsilon)^3(M + \varepsilon) + E(M + \varepsilon)^4}
\]

These inequalities yield

\[
m \geq \frac{aM^4 + bM^3 + cM^2 + dM + eM^4}{AM^4 + BM^3 + CM^2 + DM + EM},
\]

\[
M \leq \frac{aM^4 + bM^3 + cM^2 + dM^3 + eM^4}{AM^4 + BM^3 + CM^2 + DM^3 + EM}
\]

So,

\[
mM \geq M - \frac{aM^4 + bM^3 + cM^2 + dM + eM^4}{AM^4 + BM^3 + CM^2 + DM + EM},
\]

\[
mM \leq \frac{aM^4 + bM^3 + cM^2 + dM^3 + eM^4}{AM^4 + BM^3 + CM^2 + DM^3 + EM}
\]

Hence

\[
M - m \leq \frac{aM^4 + bM^3 + cM^2 + dM^3 + eM^4}{AM^4 + BM^3 + CM^2 + DM^3 + EM} - \frac{aM^4 + bM^3 + cM^2 + dM^3 + eM^4}{AM^4 + BM^3 + CM^2 + DM^3 + EM}
\]

\[
\leq 0
\]

which can be written

\[
(M - m) \leq 0
\]

\[
\leq 0
\]

Since

\[
\begin{align*}
ea(M^8 + M^6) + p_1 m n M (m^6 + M^6) + p_2 m^2 M^2 (M^4 + m^4) + p_3 m^3 M^3 (M^2 + m^2) + p_4 m^4 M^4 \\
(Am^4 + Bm^3 + Cm^2 + Dm + EM)(AM^4 + BM^3 + CM^2 + DM^3 + EM)
\end{align*}
\]

we get

\[
M \leq m.
\]

So, \( m = M = \overline{m} \).

By the same arguments, we can prove the following theorem.

\[
\square
\]

On a Second Order Rational Difference Equation 871
2.6. Theorem. Let
\[ q_1 = -Ac + Ed + (B + E)a, \]
\[ q_2 = -(A + D)e + (B + E)d + Ec - Ab + (C + B + E)a, \]
\[ q_3 = -(A + C + D)e + (B + C + E)d + (-A + B + E)c + (-A - D + E)b \]
\[ + (B + C + D + E)a, \]
\[ q_4 = (-A - B - C - D + E)c + (-A + B + C + D + E)d + (-A + B + C - D + E)c \]
\[ + (-A + B - C - D + E)b + (A + B + C + D + E)a. \]

Assume that
\[ (1) \frac{1}{e} \leq \min(\frac{1}{d}, \frac{1}{c}, \frac{1}{a}), \]
\[ (2) \frac{1}{d} \geq \max(\frac{1}{e}, \frac{1}{c}, \frac{1}{a}), \]
\[ (3) 3r_1 + r_9 \leq 0, \]
\[ (4) 2r_4 + r_8 \leq 0, \]
\[ (5) r_5 + 3r_{10} \leq 0, \]
\[ (6) \frac{2[(3r_1 + r_4 + 4r_3 + r_2 + 2r_7 + r_9 + 3r_6) + \alpha]}{(a + b + c + d + e)(A + B + C + D + E)} \leq 1, \]
\[ (7) q_1, q_2, q_3, q_4 \geq 0. \]

Then the equilibrium point \( \bar{z} = \frac{a + b + c + d + e}{a + b + c + d + e} \) of (1.1) is globally asymptotically stable.

2.7. Theorem. Assume that \( q_1, q_2, q_3, q_4 \geq 0. \) Then, Equation (1.1) has no positive solution of period two.

Proof. For the sake of contradiction, assume that there exist distinct positive real numbers \( \alpha, \beta, \alpha, \beta, \ldots \)

is a period two solution of (1.1). Then,
\[ \alpha = f(\beta, \alpha), \beta = f(\alpha, \beta). \]

Thus, we have
\[ \beta f(\beta, \alpha) = \alpha f(\alpha, \beta), \]
which implies
\[ (\beta - \alpha) \left( aE(\alpha^5 + \beta^5) + q_1\alpha\beta(\alpha^6 + \beta^6) + q_2\alpha^2\beta^2(\alpha^4 + \beta^4) + q_3\alpha^3\beta^3(\alpha^2 + \beta^2) + q_4\alpha^4\beta^4 \right) = 0. \]

Since
\[ aE(\alpha^5 + \beta^5) + q_1\alpha\beta(\alpha^6 + \beta^6) + q_2\alpha^2\beta^2(\alpha^4 + \beta^4) + q_3\alpha^3\beta^3(\alpha^2 + \beta^2) + q_4\alpha^4\beta^4 \]
\[ (A\beta^5 + B\beta^3 + C\beta^2 + D\beta + E\beta) \]
\[ (A\alpha^5 + B\alpha^3 + C\alpha^2 + D\alpha + E\alpha) \]
we get \( \alpha = \beta, \) which is a contradiction.

2.8. Remark. It follows from (1.1), when \( r_1 = \cdots = r_{10} = 0 \) that \( x_{n+1} = \alpha \) for all \( n \geq -1 \) for some constant \( \alpha. \)

3. Numerical examples

In order to illustrate our results and to support our theoretical discussions, we consider numerical examples in this section.

3.1. Example. Let \((a, b, c, d, e, A, B, C, D, E) = (3.8, 2.2, 3.3, 3.5, 1.5, 2.3, 5.3, 2.5, 5.5)\) and \((x_{-1}, x_0) = (5, 2.9).\) Then, all the conditions of Theorem 2.5 are satisfied and we have the following results:
On a Second Order Rational Difference Equation 873

\[ x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}} \]

3.2. Example. Let \((a, b, c, d, e, A, B, C, D, E) = (1.5, 2, 3.5, 3.2, 5.5, 3.8, 2.2, 3.3, 3.5)\) and \((x_1, x_0) = (0.15, 21)\). Then, all the conditions of Theorem 2.6 are satisfied and we have the following results:

| \(n\) | \(x_n - \bar{x}\) | \(n\) | \(x_n - \bar{x}\) | \(n\) | \(x_n - \bar{x}\) |
|---|---|---|---|---|
| 1 | 0.5089804983 | 21 | 0.1369202 \(10^{-3}\) | 31 | 0.1065022 \(10^{-9}\) |
| 2 | 0.219055041 | 22 | 0.13941 \(10^{-5}\) | 32 | 0.1065022 \(10^{-9}\) |
| 3 | 0.3068138396 | 23 | 0.1369202 \(10^{-3}\) | 33 | 0.1065022 \(10^{-9}\) |
| 4 | 0.1270735371 | 24 | 0.13941 \(10^{-5}\) | 34 | 0.1065022 \(10^{-9}\) |

Acknowledgement
The helpful suggestions of the anonymous referee are gratefully acknowledged.

References
[3] Cinar, C., Karatas, R. and Yalcinkaya, I. On solutions of the difference equation \(x_{n+1} = \frac{x_{n-1}}{1 + x_n x_{n-1}}\), Math. Bohem. 132 (3), 257–261, 2007.
[5] Chen, D., Li, X. and Wang, Y. Dynamics for nonlinear difference equation \(x_{n+1} = \alpha x_n + \beta x_{n-k} - \gamma x_{n-l}\), Adv. Differ. Equ., Article ID 235691, 13 pages, 2009.


[19] Yalçinkaya, I. On the difference equation \( x_{n+1} = \alpha + \frac{x_{n-2}}{x_n} \), Fasc. Math. 42, 133–139, 2009.