CONVEXITY OF INTEGRAL OPERATORS OF $p$-VALENT FUNCTIONS

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Abstract
In this paper, we consider two general $p$-valent integral operators for certain analytic functions in the unit disc $U$ and give some properties for these integral operators on some classes of univalent functions.

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1. Introduction and preliminaries
Let $A (p, n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \ (p, n \in \mathbb{N} = \{1, 2, \ldots\}),$$

which are analytic in the open disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Also $A (1, n) = A (n)$, $A (p, 1) = A (p)$ and $A (1, 1) = A$.

A function $f \in A (p, n)$ is said to be $p$-valently starlike of order $\alpha$, ($0 \leq \alpha < p$), if and only if

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \ (z \in U).$$

We denote by $S_p^*(\alpha)$ the class of all such functions. Also $S_1^*(\alpha) = S^*(\alpha)$. On the other hand, a function $f \in A (p, n)$ is said to be $p$-valently convex of order $\alpha$ ($0 \leq \alpha < p$) if and only if

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \ (z \in U).$$

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Let $C_p(\alpha)$ denote the class of all those functions which are $p$-valently convex of order $\alpha$ in $U$. Also $C(\alpha) = C(\alpha)$. A function $f \in A(p, n)$ is said to be class $R_p(\alpha)$, $(0 \leq \alpha < p)$ if and only if

\[ \Re \left\{ \frac{f'(z)}{z^{p-1}} \right\} > \alpha, \quad (z \in U). \]

Also $R_1(\alpha) = R(\alpha)$. For a function $f \in A(p, n)$ we define the following operator

\[ D^0 f(z) = f(z), \]
\[ D^1 f(z) = \frac{1}{p} z f'(z), \]
\[ : \]
\[ D^k f(z) = D \left( D^{k-1} f(z) \right), \]

where $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The differential operator $D^k$ was studied by Shenan et al. (see [14]). When $p = 1$ we get the Sălăgean differential operator (see [12]).

We note that if $f \in A(p, n)$, then

\[ D^k f(z) = z^p + \sum_{j=n+p}^{\infty} \left( \frac{1}{p} \right)^k a_j z^j, \quad (p, n \in \mathbb{N} = \{1, 2, \ldots\}) \quad (z \in U). \]

Recently, A. Alb Lupuș (see [2]) define the family $\mathcal{B}S(p, m, \mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$, $m \in \mathbb{N} \cup \{0\}$, $p, n \in \mathbb{N}$ so that it consists of functions $f \in A(p, n)$ satisfying the condition

\[ \left| \frac{D^{m+1} f(z)}{z^p} \left( \frac{z^p}{D^m f(z)} \right)^\mu p - p - \alpha \right| < \alpha, \quad (z \in U). \]

\textbf{1.1. Remark.} The family $\mathcal{B}S(p, m, \mu, \alpha)$ is a new comprehensive class of analytic functions which includes various new classes of analytic univalent functions as well as some very well-known ones. For example, $\mathcal{B}S(1, 0, 1, \alpha) \equiv S^*(\alpha)$, $\mathcal{B}S(1, 1, 1, \alpha) \equiv C(\alpha)$, $\mathcal{B}S(p, 0, 0, \alpha) \equiv R_p(\alpha)$ and $\mathcal{B}S(1, 0, 0, \alpha) \equiv R(\alpha)$.

Another interesting subclass is the special case $\mathcal{B}S(1, 0, 2, \alpha) \equiv \mathcal{B}(\alpha)$ which has been introduced by Frasin and Darus (see [7]) and also the class $\mathcal{B}S(1, 0, \mu, \alpha) \equiv \mathcal{B}(\mu, \alpha)$ which has been introduced by Frasin and Jahangiri (see [8]).

\textbf{1.2. Remark.} Let $l = (l_1, l_2, \ldots, l_n) \in \mathbb{N}_0^n$, $\delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \mathbb{R}_+^n$ for all $i = \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$. We define the following general integral operator

\[ g_{n, p}^{l, \delta}(f_1, f_2, \ldots, f_n) = A(p, n) \to A(p, n), \]
\[ g_{n, p}^{l, \delta}(f_1, f_2, \ldots, f_n) = G_{p, n, l, \delta}(z), \]
\[ G_{p, n, l, \delta}(z) = \int_0^z dt^{p-1} \prod_{i=1}^n \left( \frac{D^i f_i(t)}{t^p} \right)^{\delta_i} dt, \]

and

\[ g_{n, p}^{l, \lambda}(g_1, g_2, \ldots, g_n) = A(p, n) \to A(p, n), \]
\[ g_{n, p}^{l, \lambda}(g_1, g_2, \ldots, g_n) = G_{p, n, l, \lambda}(z), \]
\[ G_{p, n, l, \lambda}(z) = \int_0^z dt^{p-1} \prod_{i=1}^n \left( e^{D^i g_i(t)} \right)^{(p-1)} dt, \]

where $f_i, g_i \in A(p, n)$ for all $i = \{1, 2, \ldots, n\}$ and $D$ is defined by (1.5).
1.3. Remark. The integral operator (1.7) was studied and introduced by Saltık et al. (see [13]). We note that if \( l_1 = l_2 = \ldots = l_n = 0 \) for all \( i = 1, 2, \ldots, n \), then the integral operator \( F_{p,n,l,\delta}(z) \) reduces to the operator \( F_p(z) \) which was studied by Frasin (see [6]). Upon setting \( p = 1 \) in the operator (1.7), we can obtain the integral operator \( D^p F(z) \) which was studied by Breaz et al. (see [4]). For \( p = 1 \) and \( l_1 = l_2 = \ldots = l_n = 0 \) in (1.7), the integral operator \( F_{p,n,l,\delta}(z) \) reduces to the operator \( F_n(z) \) which was studied by Breaz and Breaz (see [3]). Observe that for \( p = n = 1 \), \( l_1 = 0 \) and \( \mu_1 = \mu \) we obtain the integral operator \( I_p(f)(z) \) which was studied by Pescar and Owa (see [11]), for \( \mu_1 = \mu \in [0, 1] \) the special case of the operator \( I_p(f)(z) \) was studied by Miller et al. (see [10]). For \( p = n = 1 \), \( l_1 = 0 \) and \( \mu_1 = 1 \) in (1.7), we have the Alexander integral operator \( I(f)(z) \) in [1]. For \( l_1 = l_2 = \ldots = l_n = 0 \) in (1.7), the integral operator was studied by E. Deniz (see [5]).

1.4. Remark. For \( l_1 = l_2 = \ldots = l_n = 0 \) in (1.8), the integral operator was studied by E. Deniz (see [5]). For \( p = n = 1 \) and \( l_1 = l_2 = \ldots = l_n = 0 \) in (1.8), the integral operator \( G_{p,n,l,\lambda}(z) \) was studied by Frasin in [9].

In this paper, we obtain the order of convexity of the operators \( F_{p,n,l,\delta}(z) \) and \( G_{p,n,l,\lambda}(z) \) on the class \( \mathcal{B}S(p,l,\mu,\alpha) \). As special cases, the order of convexity of the operators
\[
\int_0^z \left( \frac{f'(t)}{t} \right)^\delta \, dt \quad \text{and} \quad \int_0^z \left( e^{\varphi(t)} \right)^\lambda \, dt
\]
d are given.

In order to prove our main results, we recall the following lemma.

1.5. Lemma. (General Schwarz Lemma). Let the function \( f \) be regular in the disk \( \mathcal{U}_R \) with \( |f(z)| < M \), \( M \) fixed. If \( f \) has at \( z = 0 \) one zero with multiply \( \geq m \), then
\[
|f(z)| \leq \frac{M}{R^m} |z|^m, \quad z \in \mathcal{U}_R.
\]

Equality in the inequality (1.9) for \( z \neq 0 \) can hold only if \( f(z) = e^{i\theta} \frac{M}{R^m} |z|^m \), where \( \theta \) is constant. \( \square \)

2. Main results

2.1. Theorem. Let \( l = (l_1,l_2,\ldots,l_n) \in \mathbb{N}_0^n \), \( \delta = (\delta_1,\delta_2,\ldots,\delta_n) \in \mathbb{R}_+^n \), \( 0 \leq \alpha < p \), \( \mu \geq 0 \) and \( f_i \in A(p,n) \) be in the class \( \mathcal{B}S(p,l,\mu,\alpha) \) for all \( i = 1, 2, \ldots, n \). If \( |D^{i} f_i(z)| \leq M \), \( (M \geq 1; \quad z \in \mathcal{U}) \), then the integral operator
\[
F_{p,n,l,\delta}(z) = \int_0^z \mu^{p-1} \prod_{i=1}^n \left( \frac{D^{i} f_i(l)}{l^p} \right)^{\delta_i} \, dt
\]
is in \( C_p(\beta) \), where
\[
\beta = p \left[ 1 - \sum_{i=1}^n \delta_i ((2p - \alpha) M^{\mu-1} + 1) \right],
\]
and
\[
\sum_{i=1}^n \delta_i ((2p - \alpha) M^{\mu-1} + 1) \leq 1 \quad \text{for all} \quad i = 1, 2, \ldots, n.
\]

Proof. Define the function \( F_{p,n,l,\delta}(z) \) by
\[
F_{p,n,l,\delta}(z) = \int_0^z \mu^{p-1} \prod_{i=1}^n \left( \frac{D^{i} f_i(l)}{l^p} \right)^{\delta_i} \, dt.
\]
for \( f_i(z) \in \mathcal{B}S(p,l_i,\mu,\alpha) \). On the other hand it is easy to see that

\[
(2.2) \quad (F_{p,n,l,\delta}(z))' = pz^{p-1} \prod_{i=1}^{n} \left( \frac{D^{i+1}f_i(z)}{D^i f_i(z)} \right) \delta_i.
\]

Now, we differentiate (2.2) logarithmically and multiply by \( z \) to obtain

\[
(2.3) \quad 1 + z \frac{(F_{p,n,l,\delta}(z))''}{(F_{p,n,l,\delta}(z))'} - p = \sum_{i=1}^{n} \delta_i \left( z \frac{(D^{i+1}f_i)'(z)}{(D^i f_i)'(z)} - p \right).
\]

It follows from (2.3) and \( p(D^{i+1}f_i(z)) = z(D^i f_i(z))' \) that

\[
\left| 1 + \frac{z(F_{p,n,l,\delta}(z))''}{(F_{p,n,l,\delta}(z))'} - p \right| \leq p \sum_{i=1}^{n} \delta_i \left( \left| \frac{D^{i+1}f_i(z)}{D^i f_i(z)} \right| + 1 \right)
\]

\[
\leq p \sum_{i=1}^{n} \delta_i \left( \left| \frac{D^{i+1}f_i(z)}{D^i f_i(z)} \right| \left( \frac{z^p}{D^i f_i(z)} \right)^\mu \left| \frac{D^i f_i(z)}{z^p} \right|^{\mu-1} + 1 \right).
\]

Since \( |D^i f_i(z)| \leq M \), \((M \geq 1, z \in \mathbb{U})\) for all \( i = \{1, 2, \ldots, n\} \), applying the General Schwarz Lemma, we have

\[
\left| D^i f_i(z) \right| \leq M |z|^\mu.
\]

Therefore, from (2.4), we obtain

\[
(2.5) \quad \left| 1 + \frac{z(F_{p,n,l,\delta}(z))''}{(F_{p,n,l,\delta}(z))'} - p \right| \leq p \sum_{i=1}^{n} \delta_i \left( \left| \frac{D^{i+1}f_i(z)}{D^i f_i(z)} \right| \left( \frac{z^p}{D^i f_i(z)} \right)^\mu \left| \frac{D^i f_i(z)}{z^p} \right|^{\mu-1} + 1 \right).
\]

From (2.5) and (1.6), we see that

\[
(2.6) \quad \left| 1 + \frac{z(F_{p,n,l,\delta}(z))''}{(F_{p,n,l,\delta}(z))'} - p \right| \leq p \sum_{i=1}^{n} \delta_i \left( \left( \frac{D^{i+1}f_i(z)}{z^p} \right)^\mu \left( \frac{z^p}{D^i f_i(z)} \right)^{-\mu} \right) + 1 \right).
\]

This completes the proof. \( \square \)

**2.2. Corollary.** Let \( l = (l_1,l_2,\ldots,l_n) \in \mathbb{N}^n_0 \), \( \delta = (\delta_1,\delta_2,\ldots,\delta_n) \in \mathbb{R}^n_+ \), \( 0 \leq \alpha < p, \mu \geq 0 \) and \( f_i \in A(p,n) \) is in the class \( \mathcal{B}S(p,l_i,\mu,\alpha) \) for all \( i = \{1, 2, \ldots, n\} \). If \( |D^i f_i(z)| \leq M \), \((M \geq 1; z \in \mathbb{U})\), then the integral operator \( F_{p,n,l,\delta}(z) \) is convex in \( \mathbb{U} \) and

\[
\sum_{i=1}^{n} \delta_i = \frac{1}{(2p - \alpha) M^{\mu-1} + 1}.
\]

Letting \( p = 1, l_i = 0 \) in Theorem 2.1 for all \( i = \{1, 2, \ldots, n\} \), we have

**2.3. Corollary.** Let \( \delta = (\delta_1,\delta_2,\ldots,\delta_n) \in \mathbb{R}^n_+ \), \( \mu \geq 0 \), \( 0 \leq \alpha < 1 \) and \( f_i \in A(n) \) is in the class \( \mathcal{B}(\mu,\alpha) \) for all \( i = \{1, 2, \ldots, n\} \). If \( |f_i(z)| \leq M \), \((M \geq 1; z \in \mathbb{U})\), then the integral
operator $F_{1,n,0,δ}(z) \in \mathcal{C}(β)$ is in $\mathcal{U}$ and
\[\beta = 1 - \sum_{i=1}^{n} \delta_i \left(2 - α\right)M^{2} + 1\],
where $\sum_{i=1}^{n} \delta_i \left(2 - α\right)M^{2} + 1 \leq 1$ for all $i = \{1, 2, \ldots, n\}$. \(\Box\)

Letting $n = 1$ in Corollary 2.2, we have

2.4. Corollary. Let $δ \in \mathbb{R}^+$, $μ ≥ 0$, $0 ≤ α < 1$ and let $f \in A$ be in the class $\mathcal{B}(μ, α)$. If $|f(z)| ≤ M$, ($M ≥ 1$; $z \in \mathcal{U}$), then the integral operator $F_{1,1,0,δ}(z) = \int_0^{\frac{z}{t}} \left(\frac{t\left(1+t\right)}{1+t}\right)^{\frac{1}{3}} dt \in \mathcal{C}(β)$ is in $\mathcal{U}$, and
\[\beta = 1 - δ \left(2 - α\right)M^{2} + 1\],
where $δ \left(2 - α\right)M^{2} + 1 ≤ 1$. \(\Box\)

Letting $p = 1$, $l_i = 0$, $μ = 1$ in Theorem 2.1 for all $i = \{1, 2, \ldots, n\}$, we have

2.5. Corollary. Let $δ = (δ_1, δ_2, \ldots, δ_n) \in \mathbb{R}^+_n$, $0 ≤ α < 1$ and let $f_i \in \mathcal{A}(n)$ be in the class $\mathcal{S}^*(α)$ for all $i = \{1, 2, \ldots, n\}$. Then the integral operator $F_{1,n,0,δ}(z) \in \mathcal{C}(β)$ is in $\mathcal{U}$, where
\[\beta = 1 - \sum_{i=1}^{n} \delta_i (3 - α)\],
where $\sum_{i=1}^{n} \delta_i (3 - α) ≤ 1$ for all $i = \{1, 2, \ldots, n\}$. \(\Box\)

Letting $n = 1, δ = \frac{1}{3}$ and $α = 0$ in Corollary 2.5, we have

2.6. Corollary. Let $f \in A$ be starlike in $\mathcal{U}$. If $|f(z)| ≤ M$, ($M ≥ 1$; $z \in \mathcal{U}$), then the integral operator $F_{1,1,0,\frac{1}{3}}(z)$ is convex in $\mathcal{U}$. \(\Box\)

2.7. Remark. Letting $δ_i$ by $\frac{1}{δ_i}$, $p = 1$, $l_i = 0$ in Theorem 2.1 for all $i = \{1, 2, \ldots, n\}$ we obtain Theorem 2.1 (see [9]).

2.8. Theorem. Let $l = (l_1, l_2, \ldots, l_n) \in \mathbb{N}_0^n$, $λ \in \mathbb{R}^+$, $0 ≤ α < p$, $μ ≥ 0$ and $g_i \in \mathcal{A}(p, n)$ be in the class $\mathcal{BS}(p, l, α)$ for all $i = \{1, 2, \ldots, n\}$. If $|D^{p}g_i(z)| ≤ M$, ($M ≥ 1$; $z \in \mathcal{U}$), then the integral operator
\[(2.7) \quad \mathcal{G}_{p,n,l,λ}(z) = \int_0^{\frac{z}{t}} \prod_{i=1}^{n} \left(\frac{D^{p}g_i(t)}{\lambda}\right)^{\lambda - 1} dt,\]
is in $\mathcal{C}_p(β)$, where
\[\beta = p - \left[λn \left\{(p^2 + (1-α)p)M + (p-1)M\right\}\right],\]
and $λ ≤ \frac{p}{n\{(p^2 + (1-α)p)M + (p-1)M\}}$.

Proof. Define the function $\mathcal{G}_{p,n,l,λ}(z)$ by
\[(2.8) \quad 1 + \frac{z}{\left(\mathcal{G}_{p,n,l,λ}(z)\right)^{n}} = p = λ\sum_{i=1}^{n} \left(\frac{D^{p}g_i(z)}{x}\right)^{\lambda} - (p-1)\frac{D^{p}g_i(z)}{x}\quad z.\]
Therefore from (2.8) and \( p \left(D^{i+1}f_i\right)(z) = z \left(D^i f_i(z)\right)'\), we obtain
\[
1 + \frac{z \left(S_{p,n,l,\lambda}(z)\right)^\nu}{\left(S_{p,n,l,\lambda}(z)\right)^\nu - p} \leq \lambda \left( \sum_{i=1}^{n} \left( p \left|D^{i+1}g_i(z)\right| + (p-1) \left|\frac{D^i g_i(z)}{z_p}\right| \right) \right) \leq \lambda \left( \sum_{i=1}^{n} \left( p \left|D^{i+1}g_i(z)\right| \left(\frac{z_p}{D^i g_i(z)}\right)^\mu \left|D^i g_i(z)\right|^\mu + (p-1) \left|\frac{D^i g_i(z)}{z_p}\right| \right) \right).
\]
Applying the General Schwarz Lemma once again, we have
\[
\left|\frac{D^i g_i(z)}{z_p}\right| \leq M, \ (z \in \mathbb{U}),
\]
and hence
\[
1 + \frac{z \left(S_{p,n,l,\lambda}(z)\right)^\nu}{\left(S_{p,n,l,\lambda}(z)\right)^\nu - p} \leq \lambda \left( \sum_{i=1}^{n} \left( p \left|D^{i+1}g_i(z)\right| \left(\frac{z_p}{D^i g_i(z)}\right)^\mu \left|D^i g_i(z)\right|^\mu + (p-1) M \right) \right).
\]
Therefore from (2.9), we obtain
\[
1 + \frac{z \left(S_{p,n,l,\lambda}(z)\right)^\nu}{\left(S_{p,n,l,\lambda}(z)\right)^\nu - p} \leq \lambda \left( \sum_{i=1}^{n} \left( p \left|D^{i+1}g_i(z)\right| \left(\frac{z_p}{D^i g_i(z)}\right)^\mu - p \right) M^\mu + (p-1) M \right) \leq \lambda n \left\{ (p^2 + (1-\alpha)p) M^\mu + (p-1)M \right\} = p - \beta.
\]
This completes the proof. \( \square \)

Letting \( l_i = 0, \mu = 0 \) in Theorem 2.8 for all \( i = \{1,2,\ldots,n\} \), we have

**2.9. Corollary.** Let \( g_i \in A(n) \) be in the class \( R_p(\alpha) \), \( \lambda \in \mathbb{R}_+ \), \( 0 \leq \alpha < p \). If \( |g_i(z)| \leq M, \ (M \geq 1; \ z \in \mathbb{U}) \), then the integral operator \( S_{p,n,0,\lambda}(z) \) is in \( C_p(\beta) \) in \( \mathbb{U} \), where
\[
\beta = p - \left\{ \lambda n \left( (p^2 + (1-\alpha)p) + (p-1)M \right) \right\},
\]
and \( \lambda n \left( (p^2 + (1-\alpha)p) + (p-1)M \right) \leq p. \) \( \square \)

Letting \( n = 1, p = 1, l = 0 \) in Theorem 2.8, we have

**2.10. Corollary.** Let \( \lambda \in \mathbb{R}_+ \), \( 0 \leq \alpha < 1, \mu \geq 0 \) and let \( g \in A \) be in the class \( B(\mu,\alpha) \). If \( |g(z)| \leq M, \ (M \geq 1; \ z \in \mathbb{U}) \), then the integral operator \( S_{1,1,0,\lambda}(z) = \int_0^\infty \left(e^{g(t)}\right)^\lambda dt \) is in \( C(\beta) \) in \( \mathbb{U} \), where
\[
\beta = 1 - \lambda (2 - \alpha) M^\mu,
\]
and \( \lambda (2 - \alpha) M^\mu \leq 1. \) \( \square \)

Letting \( p = 1, l_i = 0, \mu = 1 \) in Theorem 2.8 for all \( i = \{1,2,\ldots,n\} \), we have
2.11. Corollary. Let \( g_i \in A(n) \) be in the class \( S^*(\alpha) \), \( \lambda \in \mathbb{R}^n_+ \), \( 0 \leq \alpha < 1 \), for all \( i = \{1, 2, \ldots, n\} \). If \( |g_i(z)| \leq M \), \( (M \geq 1; z \in U) \), then the integral operator \( G_{1,n,0,\lambda}(z) \) is in \( \mathcal{C}(\beta) \) in \( U \), where
\[
\beta = 1 - \lambda n(2 - \alpha)M,
\]

and \( \lambda \leq \frac{1}{n^2 - n}M, \)

Letting \( \alpha = 0 \), \( M = n = 1 \) and \( \lambda = \frac{1}{2} \) in Corollary 2.12, we have

2.12. Corollary. Let \( g \in A \) be starlike in \( U \) for all \( i = \{1, 2, \ldots, n\} \). If \( |g(z)| \leq 1 \), \( (z \in U) \), then the integral operator \( G_{1,1,0,\lambda}(z) \) is convex in \( U \). \( \square \)

References