A NOTE ON GENERALIZED LEFT $(\theta, \phi)$-DERIVATIONS IN PRIME RINGS

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Abstract

In this paper we describe generalized left $(\theta, \phi)$-derivations in prime rings, and prove that an additive mapping in a ring $R$ acting as a homomorphism or anti-homomorphism on an additive subgroup $S$ of $R$ must be either a mapping acting as a homomorphism on $S$ or a mapping acting as an anti-homomorphism on $S$, through which some related results are improved.

Keywords: Prime rings, Generalized left $(\theta, \phi)$-derivations, Mappings acting as homomorphisms, Mappings acting as anti-homomorphisms, Mappings acting as homomorphisms or anti-homomorphisms.

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1. Introduction

Let $R$ be an associative ring. Recall that an additive mapping $\mu : R \to R$ is called a derivation if $\mu(xy) = x\mu(y) + \mu(x)y$ holds for all $x, y \in R$. An additive mapping $\delta : R \to R$ is called a generalized derivation if there exists a derivation $\mu$ of $R$ such that $\delta(xy) = x\delta(y) + \mu(x)y$ holds for all $x, y \in R$. An additive mapping $\mu : R \to R$ is called a $(\theta, \phi)$-derivation if $\mu(xy) = \theta(x)\mu(y) + \mu(x)\phi(y)$ holds for all $x, y \in R$, where $\theta, \phi$ are endomorphisms of $R$. An additive mapping $\delta : R \to R$ is called a generalized $(\theta, \phi)$-derivation if there exists a $(\theta, \phi)$-derivation $\mu$ such that $\delta(xy) = \theta(x)\delta(y) + \mu(x)\phi(y)$.

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holds for all \(x, y \in R\), where \(\theta, \phi\) are endomorphisms of \(R\). Obviously the relations among these concepts can be described as the following diagram

\[
\begin{array}{c}
\text{generalized derivations} \\
\uparrow \\
\text{derivations} \\
\uparrow \\
\text{generalized } (\theta, \phi)\text{-derivations,} \\
\uparrow \\
(\theta, \phi)\text{-derivations}
\end{array}
\]

where \(\rightarrow\) denotes that the right covers the left as concepts. Generalized derivations, \((\theta, \phi)\)-derivations and generalized \((\theta, \phi)\)-derivations are studied mainly in (semi)-prime rings. For example one can search them in Brešar [6], Havala [10], Lee [11], Chang et al. [8], Cheng et al. [9], Ashraf et al. [3], and so on.

The concept of left derivations was given by Brešar and Vukman in [7]. Recall that an additive mapping \(\mu : R \rightarrow R\) is called a left derivation if \(\mu(xy) = x\mu(y) + y\mu(x)\) holds for all \(x, y \in R\). They proved that a prime ring \(R\) having a nonzero left derivation must be commutative. In fact, they stated the results in a more general form (see [7, Proposition 1.6]).

Similar to the development of the concept of derivations, the development of the concept of left derivations should have an analogue

\[
\begin{array}{c}
\text{generalized left derivations} \\
\uparrow \\
\text{left derivations} \\
\uparrow \\
\text{generalized left } (\theta, \phi)\text{-derivations,} \\
\uparrow \\
\text{left } (\theta, \phi)\text{-derivations}
\end{array}
\]

In fact, Ashraf and copartners have given the concept of left \((\theta, \phi)\)-derivations in [12, 2], and generalized left derivations in [4]. According to Ashraf et al., an additive mapping \(\delta : R \rightarrow R\) is called a generalized left derivation if there exists a left derivation \(\mu : R \rightarrow R\) such that \(\delta(xy) = x\delta(y) + y\mu(x)\) holds for all \(x, y \in R\). An additive mapping \(\mu : R \rightarrow R\) is called a left \((\theta, \phi)\)-derivation if \(\mu(xy) = \theta(x)\mu(y) + \phi(y)\mu(x)\) holds for all \(x, y \in R\), where \(\theta, \phi\) are endomorphisms of \(R\). And so it is natural to give the concept of generalized left \((\theta, \phi)\)-derivations as that an additive mapping \(\delta : R \rightarrow R\) is called a generalized left \((\theta, \phi)\)-derivation if there exists a left \((\theta, \phi)\)-derivation \(\mu\) such that \(\delta(xy) = \theta(x)\delta(y) + \phi(y)\mu(x)\) holds for all \(x, y \in R\), where \(\theta, \phi\) are endomorphisms of \(R\).

Particularly in a prime ring, for a generalized left \((\theta, \phi)\)-derivation \(\delta\) of \(R\), the left \((\theta, \phi)\)-derivation \(\mu\) such that \(\delta(xy) = \theta(x)\delta(y) + \phi(y)\mu(x)\) holds for all \(x, y \in R\) in the definition is unique. Hence generally in a prime ring, the unique \(\mu\) decided by \(\delta\) is called the associated left \((\theta, \phi)\)-derivation of \(\delta\).

Obviously in commutative rings, derivations (resp. generalized derivations, \((\theta, \phi)\)-derivations, generalized \((\theta, \phi)\)-derivations) act in accord with left derivations (resp. generalized left derivations, left \((\theta, \phi)\)-derivations, generalized left \((\theta, \phi)\)-derivations). However in a noncommutative ring, the case is quite different in general.

In this paper, firstly, we will give a note which describes the form of generalized left \((\theta, \phi)\)-derivations in prime rings under the assumption that \(\theta, \phi\) are automorphisms of \(R\) (see Theorem 2.1).

At the other hand, Bell and Kappe [5] discussed derivations acting as homomorphisms or anti-homomorphisms on a nonzero right ideal of a prime ring. Recall that an additive mapping \(f\) from a ring \(R\) into itself is said to act as a homomorphism or as an anti-homomorphism on \(S\), an additive subgroup of \(R\), if for each pair \(x, y \in S\), either \(f(xy) = f(x)f(y)\) or \(f(xy) = f(y)f(x)\) holds. Certainly the concept of mappings acting
as homomorphisms on \( S \), and the concept of mappings acting as anti-homomorphisms on \( S \) can be defined in a similar way.

Particularly, mappings acting as homomorphisms on \( S \) and mappings acting as anti-homomorphisms on \( S \) are all mappings acting as homomorphisms or anti-homomorphisms on \( S \). But one will ask whether or not these two kinds of mappings are the unique mappings acting as homomorphisms or anti-homomorphisms on an additive subgroup \( S \) of \( R \). In this paper, secondly, we will give another note which gives a firm answer on this problem (see Lemma 2.3).

Finally, in this paper, using the two results above, we will generalize the following results on left \( (\theta, \phi) \)-derivations to those on generalized left \( (\theta, \phi) \)-derivations (see Corollary 2.5 and Proposition 2.8).

1.1. Theorem. [2, Theorem 4.2] Let \( R \) be a prime ring and \( K \) a nonzero ideal of \( R \), and let \( \theta, \phi \) be automorphisms of \( R \). Suppose \( d : R \to R \) is a left \( (\theta, \phi) \)-derivation of \( R \).

1. (1) If \( d \) acts as a homomorphism on \( K \), then \( d = 0 \) on \( R \).
2. (2) If \( d \) acts as an anti-homomorphism on \( K \), then \( d = 0 \) on \( R \).

1.2. Theorem. [1, Theorem 2.7] Let \( R \) be a 2-torsion free prime ring and \( J \) a nonzero Jordan ideal and a subring of \( R \). Suppose that \( \theta, \phi \) are automorphisms of \( R \), and that \( d : R \to R \) is a left \( (\theta, \phi) \)-derivation of \( R \).

1. (1) If \( d \) acts as a homomorphism on \( J \), then \( d = 0 \) on \( R \).
2. (2) If \( d \) acts as an anti-homomorphism on \( J \), then \( d = 0 \) on \( R \).

2. Results

Now we describe the generalized left \( (\theta, \phi) \)-derivation of a prime ring \( R \) under the assumption that \( \theta, \phi \) are automorphisms of \( R \).

2.1. Theorem. Let \( R \) be a prime ring with automorphisms \( \theta, \phi \). Then a generalized left \( (\theta, \phi) \)-derivation \( \delta \) must take one of the following forms:

1. (1) There exists a left \( R \)-homomorphism \( d : R \to R \) such that \( \delta = \theta \circ d \).
2. (2) \( R \) is a commutative domain with \( \delta \) as its a generalized \( (\theta, \phi) \)-derivation.

Proof. Let \( \mu \) be the associated left \( (\theta, \phi) \)-derivation of \( \delta \).

Firstly, we consider the case that \( \mu = 0 \). Then \( \delta(xy) = \theta(x)\delta(y) \) holds for all \( x, y \in R \). Let \( d = \theta^{-1} \circ \delta \), we can obtain \( \delta = \theta \circ d \) with \( d \) a left \( R \)-homomorphism from \( R \) into itself, which is the first case.

Finally, we consider the case that \( \mu \neq 0 \). Then for all \( x, y, z \in R \), we have

\[
\delta(xyz) = \delta((xy)z) = \theta(xy)\delta(z) + \phi(z)\mu(xy)
= \theta(x)\theta(y)\delta(z) + \phi(z)\theta(x)\mu(y) + \phi(z)\phi(y)\mu(x).
\]

At the other hand, for all \( x, y, z \in R \), we have

\[
\delta(xyz) = \delta(x(yz)) = \theta(x)\delta(yz) + \phi(yz)\mu(x)
= \theta(x)\theta(y)\delta(z) + \theta(x)\phi(z)\mu(y) + \phi(y)\phi(z)\mu(x).
\]

So for all \( x, y, z \in R \), we have

\[
\theta(x)\theta(y)\delta(z) + \theta(x)\phi(z)\mu(y) + \phi(y)\phi(z)\mu(x) = 0.
\]

Setting \( z = y \) in (2.1), we have that

\[
\theta(x)\phi(y)\mu(y) = 0
\]
holds for all \( x, y \in R \). Setting \( x = xz \) in (2.2), we have
\[
\theta(x)[\theta(z), \phi(y)]\mu(y) + [\theta(x), \phi(y)]\theta(z)\mu(y) = 0
\]
holds for all \( x, y, z \in R \). By (2.2), for all \( x, y, w \in R \), we have \([\theta(x), \phi(y)]\mu(y) = 0\).
Hence for each \( y \in R \), either \( \mu(y) = 0 \) or \( \phi(y) \in Z(R) \) since \( R \) is a prime ring. That is
\[
\{ y \in R \mid \phi(y) \in Z(R) \} \cup \{ y \in R \mid \mu(y) = 0 \} = R.
\]
Hence either \( \{ y \in R \mid \phi(y) \in Z(R) \} = R \) or \( \{ y \in R \mid \mu(y) = 0 \} = R \).
Since \( \mu \neq 0 \), we have \( \{ y \in R \mid \phi(y) \in Z(R) \} = R \). Then \( R \) is a commutative domain which completes the proof. □

By Theorem 2.1, we give the form of the left \((\theta, \phi)\)-derivation of a prime ring \( R \) under the assumption that \( \theta, \phi \) are automorphisms of \( R \).

2.2. Corollary. Let \( R \) be a prime ring with automorphisms \( \theta, \phi \). Then \( \mu \) is a nonzero left \((\theta, \phi)\)-derivation of \( R \) if and only if \( R \) is a commutative domain with \( \mu \) as its a nonzero \((\theta, \phi)\)-derivation. □

Now we give another note on mappings acting as homomorphisms or anti-homomorphisms on an additive subgroup of a ring.

2.3. Lemma. Let \( R \) be a ring with \( S \) an additive subgroup. Let \( f : R \rightarrow R \) be an additive mapping. Then \( f \) acts as a homomorphism or an anti-homomorphism on \( S \) if and only if either \( f \) acts as a homomorphism on \( S \) or \( f \) acts as an anti-homomorphism on \( S \).

Proof. We will deal with the only if part, for the other part is obvious. For each \( s \in S \), let \( H_s = \{ x \in S \mid f(sx) = f(s)f(x) \} \) and \( H'_s = \{ x \in S \mid f(sx) = f(x)f(s) \} \). Obviously \( H_s \) and \( H'_s \) are all subgroups of \( S \), and \( H_s \cup H'_s = S \). So either \( H_s = S \) or \( H'_s = S \).
Let \( H = \{ s \in S \mid H_s = S \} \) and \( H' = \{ s \in S \mid H'_s = S \} \). Obviously \( H, H' \) are all subgroups of \( S \) and \( H \cup H' = S \). So either \( H = S \) or \( H' = S \) which completes the proof. □

Note that Theorem 1.1 and 1.2 can be stated in a new form by an application of Lemma 2.3, on which we will not say more. Now we give an equivalent condition under which a generalized left \((\theta, \phi)\)-derivation \( \delta \) of a prime ring \( R \) with the associated \((\theta, \phi)\)-derivation \( \mu \) has the property that \( \mu \neq 0 \) acts as a homomorphism or an anti-homomorphism on a nonzero subring of \( R \).

2.4. Theorem. Let \( R \) be a prime ring. Let \( \delta \) be a generalized left \((\theta, \phi)\)-derivation of \( R \) with associated left \((\theta, \phi)\)-derivation \( \mu \) such that \( \mu \neq 0 \), where \( \theta, \phi \) are automorphisms of \( R \). Then \( \delta \) acts as a homomorphism or an anti-homomorphism on \( S \), a nonzero subring of \( R \), if and only if one of the following holds:

1. \( \delta = 0 \) on \( S \).
2. \( \delta = \theta \) on \( S \) and \( \mu = 0 \) on \( S \).
3. \( \delta = \phi \) on \( S \) and \( \mu = \phi - \theta \) on \( S \).

Proof. By Theorem 2.1, \( R \) is a commutative domain with \( \delta \) as its a generalized \((\theta, \phi)\)-derivation since \( \mu \neq 0 \). We will only prove the only if part, the proof for the other part is obvious. Assume that \( \delta \neq 0 \) on \( S \). Then for all \( x, y \in S \), we have
\[
\delta(xy) = \theta(x)\delta(y) + \phi(y)\mu(x) = \delta(x)\delta(y)
\]
since \( R \) is commutative. Then
\[
(\delta - \theta)(x)\delta(y) = (\mu(x)\phi(y)
\]
holds for all \( x, y \in S \). Setting \( x = xz \) in (2.3), then for all \( x, y, z \in S \), we have

\[
(\delta - \theta)(xz)\delta(y) = \mu(xz)\phi(y).
\]

That is

\[
(\theta(x)\delta(z) + \phi(z)\mu(x) - \theta(x)\theta(z))\delta(y) = \theta(x)\mu(z)\phi(y) + \phi(z)\mu(x)\phi(y)
\]

holds for all \( x, y, z \in S \). Then by (2.3) for all \( x, y, z \in S \), we have \( \phi(z)\mu(x)(\delta - \phi)(y) = 0 \).

Since \( S \neq 0 \), \( \mu(S)(\delta - \phi)(S) = 0 \). Hence either \( \mu = 0 \) on \( S \) or \( \delta = \phi \) on \( S \). When \( \mu = 0 \) on \( S \), we obtain \( \delta = \theta \) on \( S \) from (2.3). When \( \mu \neq 0 \) on \( S \), we have \( \delta = \phi \) on \( S \). Then we obtain \( \mu = \phi = \theta \) on \( S \) from (2.3). \( \square \)

Particularly when \( S \) is either a nonzero ideal of a prime ring or a nonzero Jordan ideal and subring of a 2-torsionfree prime ring \( R \) in Theorem 2.4, we have

2.5. Corollary. Let \( S \) be either a nonzero ideal of a prime ring \( R \) or a nonzero Jordan ideal and subring of a 2-torsionfree prime ring \( R \). Let \( \delta \) be a generalized left \((\theta, \phi)\)-derivation of \( R \) with the associated left \((\theta, \phi)\)-derivation \( \mu \) such that \( \mu \neq 0 \), where \( \theta, \phi \) are automorphisms of \( R \). Then \( \delta \) acts as a homomorphism or an anti-homomorphism on \( S \) if and only if \( \delta = \phi \) and \( \mu = \phi - \theta \).

Proof. When \( S \) is a nonzero ideal of a prime ring \( R \), we consider the three cases in Theorem 2.4 separately. Firstly, if \( \delta = 0 \) on \( S \), then for all \( s \in S \) and for all \( r \in R \), we have that

\[
0 = \delta(rs) = \theta(r)\delta(s) + \phi(s)\mu(r) = \phi(s)\mu(r).
\]

Then since \( S \neq 0 \), we have \( \mu = 0 \), which contradicts \( \mu \neq 0 \). Secondly, if \( \delta = \theta \) and \( \mu = 0 \) on \( S \), then for all \( s \in S \) and for all \( r \in R \), we have that

\[
\theta(r)\theta(s) = \theta(rs) = \delta(rs) = \theta(r)\delta(s) + \phi(s)\mu(r) = \theta(r)\theta(s) + \phi(s)\mu(r).
\]

Then \( \phi(s)\mu(r) = 0 \) holds for all \( s \in S \) and for all \( r \in R \), which shows that \( \mu = 0 \), a contradiction. Thirdly, if \( \delta = \phi \) and \( \mu = \phi - \theta \) on \( S \), then for all \( s \in S \) and for all \( r \in R \), we have that

\[
\phi(r)\phi(s) = \phi(rs) = \delta(rs) = \theta(r)\delta(s) + \phi(s)\mu(r) = \phi(s)(\mu(r) + \theta(r)),
\]

which shows that \( \mu(r) = (\phi - \theta)(r) \) holds for all \( r \in R \) since \( S \neq 0 \). On the other hand, for all \( s \in S \) and for all \( r \in R \), we have that

\[
\phi(s)\phi(r) = \phi(sr) = \delta(sr) = \theta(s)\delta(r) + \phi(r)\mu(s) = \theta(s)\delta(r) + \phi(r)(\phi(s) - \theta(s)).
\]

Then for all \( s \in S \) and for all \( r \in R \), we have that \( \theta(s)(\delta(r) - \phi(r)) = 0 \), which shows that \( \delta = \phi \) since \( S \neq 0 \).

When \( S \) is a nonzero Jordan ideal and subring of 2-torsionfree prime ring \( R \), noting that \( (2r)s = st + rs \in S \) for all \( r \in R \) and for all \( s \in S \) since \( R \) is commutative, in a similar way to the ideal case, we have that either \( 2\mu(x) = 0 \) holds for all \( x \in R \) or \( 2(\delta - \phi)(x) = 2(\phi - \theta - \mu)(x) = 0 \) holds for all \( x \in R \). Hence the conclusion is obtained since \( R \) is 2-torsionfree. \( \square \)

The left \((\theta, \phi)\)-derivation version of Theorem 2.4 and Corollary 2.5 can be obtained immediately.

2.6. Corollary. Let \( R \) be a prime ring . Let \( \mu \) be a left \((\theta, \phi)\)-derivation of \( R \), where \( \theta, \phi \) are automorphisms of \( R \). Then \( \mu \) acts as a homomorphism or an anti-homomorphism on \( S \), a nonzero subring of \( R \), if and only if \( \mu = 0 \) on \( S \). \( \square \)
2.7. Corollary. Let $S$ be either a nonzero ideal of a prime ring $R$ or a nonzero Jordan ideal and subring of a 2-torsionfree prime ring $R$. Let $\mu$ be a left $(\theta, \phi)$-derivation of $R$, where $\theta, \phi$ are automorphisms of $R$. Then $\mu$ acts as a homomorphism or an anti-homomorphism on $S$ if and only if $\mu = 0$.

For completeness, we discuss Corollary 2.5 further when $\mu = 0$.

2.8. Proposition. Let $S$ be either a nonzero ideal of a prime ring $R$ or a nonzero Jordan ideal and subring of a 2-torsionfree prime ring $R$. Let $\delta$ be a generalized left $(\theta, \phi)$-derivation of $R$ with the associated left $(\theta, \phi)$-derivation $\mu$ such that $\mu = 0$, where $\theta, \phi$ are automorphisms of $R$. Then $\delta$ acts as a homomorphism or an anti-homomorphism on $S$ if and only if either $\delta = \theta$ or $\delta = 0$.

Proof. By Theorem 2.1, there exists a left $R$-homomorphism $d : R \to R$ such that $\delta = \theta \circ d$. And it is easy to see that $d$ also acts as a homomorphism or an anti-homomorphism on $S$.

Firstly, We consider the case that $S$ is a nonzero ideal of a prime ring $R$. When $d$ acts as a homomorphism on $S$, then for all $s, t \in S$ and for all $x, y, z \in R$, we have

$$d(sxyz) = sxyzd(z) = d(s)xyzd = sxdz = d(z)sxd.$$

Then $s(d(x) - x)yzd = 0$ holds for all $s, t \in S$ and for all $x, y, z \in R$. Hence either $S(d(x) - x) = 0$ holds for all $x \in R$ or $Sd(z) = 0$ holds for all $z \in R$. And so either $d = 1_R$ or $d = 0$. Thus either $\delta = \theta$ or $\delta = 0$.

When $d$ acts as an anti-homomorphism on $S$, then for all $s, t \in S$, we have $d(st) = sd(t) = d(t)s$. For all $x \in R$, set $s = xs$ in the above formula, we have that

$$d(t)xd(s) = d(t)d(xs) = d((xs)t) = xsd(t) = xsd(t)d(s)$$

holds for all $s, t \in S$ and for all $x \in R$. That is $[d(t), x]d(s) = 0$ holds for all $s, t \in S$ and for all $x \in R$. Then for all $y \in R$, replacing $x$ by $xy$ in $[d(t), x]d(s) = 0$, we have that $[d(t), xy]d(s) = 0$ holds for all $s, t \in S$ and for all $x, y \in R$, which shows that $d(S) \subseteq Z(R)$. Hence $d$ acts as an anti-homomorphism on $S$ which has been dealt with.

Secondly, we consider the case that $S$ is a nonzero Jordan ideal and subring of a 2-torsionfree prime ring $R$. Note the following two facts:

1. For all $s, t \in S$ and for all $x \in R$, $2sxt \in S$.
2. For any $a \in R$, either $Sa = 0$ or $aS = 0$ implies $a = 0$.

For all $s, t \in S$ and for all $x \in R$, we have

$$2sxt + (st)x + x(st) = s(tx + xt) + (sx + xs)t \in S.$$

By $(st)x + x(st) \in S$, the first fact is proved. If $Sa = 0$, then $(sx + xs)a = 0$ for all $s \in S$ and all $x \in R$. Then $SRa = 0$ implies $a = 0$ since $S \neq 0$, which proves the second fact.

If $d$ acts as a homomorphism on $S$, then for all $r, s, t \in S$ and for all $x \in R$, we have $d(2sxt) = d(r)d(2sxt) = r(d(2sxt))$. Then $2(d(r) - r)Srd(t) = 0$ holds for all $r, t \in S$. Since $R$ is 2-torsionfree, either $d(r) = r$ holds for all $r \in S$ or $d = 0$ on $S$. Then for all $x \in R$ and for all $r \in S$, we have either that

$$xs + sx = d(xs + xs) = xd(s) + sdx = sxs + sdx$$

or that $0 = d(xs + x) = xd(s) = sdx = d(x)$. Then either $S(d(x) - x) = 0$ holds for all $x \in R$ or $Sd(x) = 0$ holds for all $x \in R$, which proves the conclusion. If $d$ acts as an anti-homomorphism on $S$, then for all $s, t \in S$, we have $d(st) = sd(t) = d(t)d(s)$. For all $r \in S$ and for all $x \in R$, setting $s = 2rxs$ in $sd(t) = d(t)d(s)$, we have $2rxd(t) = d(t)(2rxd(s))$. At the other hand, multiplying $sd(t) = d(t)d(s)$ by $2r$ from the left hand side, we have $2rxd(t) = 2rxd(t)d(s)$ for all $r, s, t \in S$ and for all $x \in R$. Hence $2[r, d(t)]d(s) = 0$.
holds for all \( r, s, t \in S \) and for all \( x \in R \). And so for all \( r, r', s, t \in S \) and for all \( x, x' \in R \), we have \( 2[r, d(t)]r'x'd(s) = 0 \). Then \( [r, d(S)]SRd(S) = 0 \) holds for all \( r \in S \) and for all \( x \in R \) since \( R \) is 2-torsionfree. Hence \( [r, d(S)] = 0 \) holds for all \( r \in S \) and for all \( x \in R \). For all \( y \in R \), setting \( x = xy \), we have that \( SR[R, d(S)] = 0 \). So \( d(S) \subseteq Z(R) \).

This shows that \( d \) acts as a homomorphism on \( S \) which we have dealt with. \( \square \)

Now we give two examples in order to show that for the Jordan ideal case the condition that \( R \) is 2-torsionfree is necessary in Corollary 2.5, Corollary 2.7 and Proposition 2.8.

2.9. Example. Let \( R = \mathbb{Z}_2[x, y] \) and
\[
S = \{ f(x, y) \in R \mid f(x, y) \text{ is a symmetrical polynomial} \}.
\]
It is easy to see that \( S \) is a nonzero Jordan ideal and subring of 2-torsion prime ring \( R \).

Let \( \theta = 1_R \) and \( \phi : R \to R \) such that \( \phi(f(x, y)) = f(y, x) \) for all \( f(x, y) \in R \). It can be checked that \( \phi \) is an automorphism of \( R \). Let \( \mu = \phi - \theta \), then \( \mu \) is a nonzero left \((\theta, \phi)\)-derivation of \( R \) and \( \mu \neq \phi \). However \( \mu(S) = 0 \) shows that \( \mu \) acts as a homomorphism or an anti-homomorphism on \( S \). This shows that for the Jordan ideal case the condition that \( R \) is 2-torsionfree is necessary in Corollary 2.5 and 2.7.

2.10. Example. Let \( R = M_2(\mathbb{Z}_2) \) and \( S = \{0, I_2 \} \subseteq R \). Then \( S \) is a nonzero Jordan ideal and subring of a 2-torsion prime ring \( R \). Let \( \theta = 1_R \) and \( f : R \to R \) such that \( f(x) = xe_{11} \) for all \( x \in R \). It is easy to see that \( f \) acts as a homomorphism on \( S \). However \( f \neq 1_R = \theta \) and \( f \neq 0 \). This shows that for the Jordan ideal case the condition that \( R \) is 2-torsionfree is necessary in Proposition 2.8.

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