SOME NEW HADAMARD TYPE INEQUALITIES FOR CO-ORDINATED m-CONVEX AND (α, m)-CONVEX FUNCTIONS

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Abstract
In this paper, we establish some new Hermite-Hadamard type inequalities for m-convex and (α, m)-convex functions of 2-variables on the co-ordinates.

Keywords: m-convex function, (α, m)-convex function, co-ordinated convex mapping, Hermite-Hadamard inequality.

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1. Introduction
Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex mapping defined on the interval \( I \) of real numbers, and \( a, b \in I \) with \( a < b \). The following double inequality is well known in the literature as the Hermite-Hadamard inequality [5]:
\[
f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a)+f(b)}{2}.
\]
In [8], the notion of m-convexity was introduced by G. Toader as the following:

1.1. Definition. The function \( f : [0, b] \rightarrow \mathbb{R} \) is said to be m-convex, where \( m \in [0, 1] \), if we have
\[
f \left( tx + m (1-t) y \right) \leq tf(x) + m(1-t)f(y)
\]
for all \( x, y \in [0, b] \) and \( t \in [0, 1] \). We say that \( f \) is m-concave if \( -f \) is m-convex.
Denote by \( K_m(b) \) the class of all \( m \)-convex functions on \([0,b]\) for which \( f(0) \leq 0 \). Obviously, if we choose \( m = 1 \), Definition 1.1 recaptures the concept of standard convex functions on \([0,b]\).

In [6], S. S. Dragomir and G. Toader proved the following Hadamard type inequalities for \( m \)-convex functions.

1.2. Theorem. Let \( f : [0, \infty) \to \mathbb{R} \) be an \( m \)-convex function with \( m \in (0,1] \). If \( 0 \leq a < b < \infty \) and \( f \in L_1[a,b] \), then the following inequality holds:

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \min \left\{ \frac{f(a) + mf(b)}{2}, \frac{f(b) + mf(a)}{2} \right\}.
\]

Some generalizations of this result can be found in [2, 3].

1.3. Theorem. Let \( f : [0, \infty) \to \mathbb{R} \) be an \( m \)-convex differentiable function with \( m \in (0,1] \). Then for all \( 0 \leq a < b \) the following inequality holds:

\[
\frac{f(mb) - \frac{b-a}{2} f'(mb)}{m} \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{(b-\alpha m) f(b) - (a-\alpha m) f(a)}{2 (b-a)}.
\]

Also, in [5], Dragomir and Pearce proved the following Hadamard type inequality for \( m \)-convex functions.

1.4. Theorem. Let \( f : [0, \infty) \to \mathbb{R} \) be an \( m \)-convex function with \( m \in (0,1] \) and \( 0 \leq a < b \). If \( f \in L_1[a,b] \), then one has the inequality:

\[
f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx + mf \left( \frac{a+b}{2} \right) dx.
\]

In [7], the definition of \((\alpha,m)\)-convexity was introduced by V.G. Miheşan as the following:

1.5. Definition. The function \( f : [0, b] \to \mathbb{R} \), \( b > 0 \), is said to be \((\alpha,m)\)-convex, where \((\alpha,m) \in [0,1]^2\), if we have

\[
f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)
\]

for all \( x, y \in [0,b] \) and \( t \in [0,1] \).

Denote by \( K_{\alpha,m}(b) \) the class of all \((\alpha,m)\)-convex functions on \([0,b]\) for which \( f(0) \leq 0 \). If we take \((\alpha,m) = (1,m)\), it can be easily seen that \((\alpha,m)\)-convexity reduces to \( m \)-convexity, and for \((\alpha,m) = (1,1)\), \((\alpha,m)\)-convexity reduces to the usual concept of convexity defined on \([0,b] \), \( b > 0 \).

In [8], E. Set, M. Sardari, M. E. Özdemir and J. Rroin proved the following Hadamard type inequalities for \((\alpha,m)\)-convex functions.

1.6. Theorem. Let \( f : [0, \infty) \to \mathbb{R} \) be an \((\alpha,m)\)-convex function with \((\alpha,m) \in (0,1]^2\). If \( 0 \leq a < b < \infty \) and \( f \in L_1[a,b] \cap L_1 \left[ \frac{a}{m}, \frac{b}{m} \right] \), then the following inequality holds:

\[
f \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x) + m(2^\alpha - 1)f \left( \frac{x}{m} \right) dx.
\]

1.7. Theorem. Let \( f : [0, \infty) \to \mathbb{R} \) be an \((\alpha,m)\)-convex function with \((\alpha,m) \in (0,1]^2\). If \( 0 \leq a < b < \infty \) and \( f \in L_1[a,b] \), then the following inequality holds:

\[
\frac{1}{b-a} \int_a^b f(x) dx \leq \min \left\{ \frac{f(a) + mof \left( \frac{b}{m} \right)}{\alpha + 1}, \frac{f(b) + mof \left( \frac{a}{m} \right)}{\alpha + 1} \right\}.
\]
1.8. Theorem. Let \( f : [0, \infty) \to \mathbb{R} \) be an \((\alpha, m)\)-convex function with \((\alpha, m) \in (0, 1)^2\). If \(0 < a < b < \infty\) and \( f \in L_1[a, b] \), then the following inequality holds:

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{1}{2} \left[ f(a) + f(b) + m \alpha f \left( \frac{a+b}{2} \right) \right].
\]

Let us now consider a bidimensional interval \( \Delta := [a, b] \times [c, d] \) in \( \mathbb{R}^2 \), with \( a < b \) and \( c < d \). A function \( f : \Delta \to \mathbb{R} \) is said to be convex on \( \Delta \) if the following inequality:

\[
f(tx + (1 - t) z, ty + (1 - t) w) \leq tf(x, y) + (1 - t)f(z, w)
\]

holds, for all \((x, y), (z, w) \in \Delta \) and \( t \in [0, 1] \). A function \( f : \Delta \to \mathbb{R} \) is said to be convex on the co-ordinates on \( \Delta \) if the partial mappings \( f_\alpha : [a, b] \to \mathbb{R} \), \( f_\alpha (u) = f(u, y) \) and \( f_\alpha : [c, d] \to \mathbb{R} \), \( f_\alpha (v) = f(x, v) \) are convex where defined for all \( x \in [a, b] \) and \( y \in [c, d] \) (see [5, p. 317]).

Also, in [4], Dragomir proved the following similar inequalities of Hadamard’s type for a co-ordinated convex mapping on a rectangle in the plane \( \mathbb{R}^2 \).

1.9. Theorem. Suppose that \( f : \Delta \to \mathbb{R} \) is co-ordinated convex on \( \Delta \). Then one has the inequalities:

\[
f \left( \frac{a+b}{2}, \frac{c+d}{2} \right)
\leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f \left( x, \frac{c+d}{2} \right) \, dx + \frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) \, dy \right]
\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) \, dx \, dy
\leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c) \, dx + \frac{1}{b-a} \int_a^b f(x, d) \, dx + \frac{1}{d-c} \int_c^d f(a, y) \, dy + \frac{1}{d-c} \int_c^d f(b, y) \, dy \right]
\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}
\]

The above inequalities are sharp. □

For co-ordinated s-convex functions, another version of this result can be found in [1].

The main purpose of this paper is to establish new Hadamard-type inequalities for functions of 2-variables which are \( m \)-convex or \((\alpha, m)\)-convex on the co-ordinates.

2. Inequalities for co-ordinated \( m \)-convex functions

Firstly, we can define co-ordinated \( m \)-convex functions as follows:

2.1. Definition. Consider the bidimensional interval \( \Delta := [0, b] \times [0, d] \) in \( [0, \infty)^2 \). The mapping \( f : \Delta \to \mathbb{R} \) is \( m \)-convex on \( \Delta \) if

\[
f(tx + (1 - t) z, ty + m (1 - t) w) \leq tf(x, y) + m (1 - t) f(z, w)
\]

holds for all \((x, y), (z, w) \in \Delta \) with \( t \in [0, 1] \), \( b, d > 0 \), and for some fixed \( m \in [0, 1] \).

A function \( f : \Delta \to \mathbb{R} \) which is \( m \)-convex on \( \Delta \) is called co-ordinated \( m \)-convex on \( \Delta \) if the partial mappings

\[
f_\alpha : [0, b] \to \mathbb{R}, \ f_\alpha (u) = f(u, y)
\]
Suppose that

\[ f_x : [0, d] \to \mathbb{R}, \quad f_x(v) = f(x, v) \]

are m-convex for all \( y \in [0, d] \) and \( x \in [0, b] \) with \( b, d > 0 \), and for some fixed \( m \in (0, 1] \).

We also need the following Lemma for our main results.

**2.2. Lemma.** Every m-convex mapping \( f : \Delta \subset [0, \infty)^2 \to \mathbb{R} \) is m-convex on the co-ordinates, where \( \Delta = [0, b] \times [0, d] \) and \( m \in [0, 1] \).

**Proof.** Suppose that \( f : \Delta = [0, b] \times [0, d] \to \mathbb{R} \) is m-convex on \( \Delta \). Consider the function

\[ f_x : [0, d] \to \mathbb{R}, \quad f_x(v) = f(x, v), \quad (x \in [0, b]). \]

Then for \( t, m \in [0, 1] \) and \( v_1, v_2 \in [0, d] \), we have

\[ f_x(tv_1 + m(1 - t)v_2) = f(tx + (1 - t)x, tv_1 + m(1 - t)v_2) \]
\[ \leq tf_x(v_1) + m(1 - t)f_x(v_2) \]
\[ = tf_x(v_1) + m(1 - t)f_x(v_2). \]

Therefore, \( f_x(v) = f(x, v) \) is m-convex on \([0, d] \). The fact that \( f_y : [0, b] \to \mathbb{R}, \quad f_y(u) = f(u, y) \) is also m-convex on \([0, b] \) for all \( y \in [0, d] \) goes likewise, and we shall omit the details. \( \square \)

**2.3. Theorem.** Suppose that \( f : \Delta = [0, b] \times [0, d] \to \mathbb{R} \) is an m-convex function on the co-ordinates on \( \Delta \). If \( 0 \leq a < b < \infty \) and \( 0 \leq c < d < \infty \) with \( m \in (0, 1] \), then one has the inequality:

\[
\frac{1}{(d - c)(b - a)} \int_c^d \int_a^b f(x, y) \, dx \, dy \\
\leq \frac{1}{4(b - a)} \min \{v_1, v_2\} + \frac{1}{4(d - c)} \min \{v_3, v_4\},
\]

where

\[
v_1 = \int_a^b f(x, c) \, dx + m \int_a^b f \left( x, \frac{d}{m} \right) \, dx
\]
\[
v_2 = \int_a^b f(x, d) \, dx + m \int_a^b f \left( x, \frac{c}{m} \right) \, dx
\]
\[
v_3 = \int_c^d f(a, y) \, dy + m \int_c^d f \left( \frac{b}{m}, y \right) \, dy
\]
\[
v_4 = \int_c^d f(b, y) \, dy + m \int_c^d f \left( \frac{a}{m}, y \right) \, dy.
\]

**Proof.** Since \( f : \Delta \to \mathbb{R} \) is co-ordinated m-convex on \( \Delta \) it follows that the mapping

\[ g_x : [0, d] \to \mathbb{R}, \quad g_x(y) = f(x, y) \]

is m-convex on \([0, d] \) for all \( x \in [0, b] \). Then by the inequality (1.1) one has:

\[
\frac{1}{d - c} \int_c^d g_x(y) \, dy \leq \min \left\{ \frac{g_x(c) + mg_x \left( \frac{d}{m} \right)}{2}, \frac{g_x(d) + mg_x \left( \frac{c}{m} \right)}{2} \right\},
\]

or

\[
\frac{1}{d - c} \int_c^d f(x, y) \, dy \leq \min \left\{ \frac{f(x, c) + mf \left( x, \frac{d}{m} \right)}{2}, \frac{f(x, d) + mf \left( x, \frac{c}{m} \right)}{2} \right\},
\]

where \( 0 \leq c < d < \infty \) and \( m \in (0, 1] \).
Dividing both sides by \((b - a)\) and integrating this inequality over \([a, b]\) with respect to \(x\), we have
\[
\frac{1}{(b - a) (d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \\
\leq \min \left\{ \frac{1}{2(b - a)} \int_a^b f(x, c) \, dx + \frac{m}{2(b - a)} \int_a^b f\left(x, \frac{d}{m}\right) \, dx, \right. \\
\frac{1}{2(b - a)} \int_a^b f(x, d) \, dx + \frac{m}{2(b - a)} \int_a^b f\left(x, \frac{c}{m}\right) \, dx \right\}
\]
(2.2)
\[
= \frac{1}{2(b - a)} \min \left\{ \int_a^b f(x, c) \, dx + m \int_a^b f\left(x, \frac{d}{m}\right) \, dx, \right. \\
\int_a^b f(x, d) \, dx + m \int_a^b f\left(x, \frac{c}{m}\right) \, dx \right\}
\]
where \(0 \leq a < b < \infty\).

By a similar argument applied to the mapping \(g_y : [0, b] \to \mathbb{R}, g_y(x) = f(x, y)\) with \(0 \leq a < b < \infty\), we get
\[
\frac{1}{(d - c) (b - a)} \int_c^d \int_a^b f(x, y) \, dx \, dy \\
\leq \frac{1}{2(d - c)} \min \left\{ \int_c^d f(a, y) \, dy + m \int_c^d f\left(\frac{b}{m}, y\right) \, dy, \right. \\
\int_c^d f(b, y) \, dy + m \int_c^d f\left(\frac{a}{m}, y\right) \, dy \right\}
\]
(2.3)
Summing the inequalities (2.2) and (2.3), we get the inequality (2.1). \(\blacksquare\)

2.4. Corollary. With the above assumptions, and provided that the partial mappings
\[f_y : [0, b] \to \mathbb{R}, f_y(u) = f(u, y)\]
and
\[f_x : [0, d] \to \mathbb{R}, f_x(v) = f(x, v)\]
are differentiable on \((0, b)\) and \((0, d)\), respectively, we have the inequalities
\[
\frac{1}{(b - a) (d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \\
\leq \frac{1}{4(b - a)} \min \left\{ (b - ma) \left[ f(b, c) + mf\left(b, \frac{d}{m}\right) \right] \right. \\
- (a - mb) \left[ f(a, c) + mf\left(a, \frac{d}{m}\right) \right], \\
(b - ma) \left[ f(b, d) + mf\left(b, \frac{c}{m}\right) \right] \\
- (a - mb) \left[ f(a, d) + mf\left(a, \frac{c}{m}\right) \right], \\
\right. \right\}
\]
(2.4)
and
Proof. Since the partial mappings 
\[
f_x : [0,d] \to \mathbb{R}, \quad f_x(v) = f(x,v)
\]
are differentiable on \([0,d]\), by the inequality (1.2) we have
\[
\frac{1}{(b-a)} \int_a^b f(x,c) \, dx \leq \left( \frac{b-ma}{2(b-a)} \right) f(b,c) - \left( \frac{a-mb}{2(b-a)} \right) f(a,c),
\]
\[
\frac{1}{(b-a)} \int_a^b f(x,d) \, dx \leq \left( \frac{b-ma}{2(b-a)} \right) f(b,d) - \left( \frac{a-mb}{2(b-a)} \right) f(a,d), \quad \text{and}
\]
\[
\frac{1}{(b-a)} \int_a^b f(x,d) \, dx \leq \left( \frac{b-ma}{2(b-a)} \right) f(b,m) - \left( \frac{a-mb}{2(b-a)} \right) f(a,m).
\]
Hence, using (2.2), we get the inequality (2.4).

Analogously, Since the partial mappings 
\[
f_y : [0,b] \to \mathbb{R}, \quad f_y(u) = f(u,y)
\]
are differentiable on \([0,b]\), by the inequality (2.3), we get the inequality (2.5). The proof is completed.

2.5. Remark. Choosing \(m = 1\) in (2.4) or (2.5), we get the relationship between the third and fourth inequalities in (1.7).

2.6. Theorem. Suppose that \(f : \Delta = [0,b] \times [0,d] \to \mathbb{R}\) is an \(m\)-convex function on the co-ordinates on \(\Delta\). If \(0 \leq a < b < \infty\) and \(0 \leq c < d < \infty\), \(m \in (0,1]\) with \(f_x \in L_1[0,d]\) and \(f_y \in L_1[0,b]\), then one has the inequality:
\[
\frac{1}{b-a} \int_a^b f \left( x, \frac{c+d}{2} \right) \, dx + \frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) \, dy
\]
\[
\leq \frac{1}{(b-a)(d-c)} \left[ \int_a^b \int_c^d f(x,y) + mf \left( x, \frac{a+b}{2} \right) \, dy \, dx \right.
\]
\[
+ \int_c^d \int_a^b f(x,y) + mf \left( \frac{x}{m}, y \right) \, dx \, dy \right].
\]
Proof. Since \(f : \Delta \to \mathbb{R}\) is co-ordinated \(m\)-convex on \(\Delta\) it follows that the mapping 
\(g_x : [0,d] \to \mathbb{R}, \quad g_x(y) = f(x,y)\) is \(m\)-convex on \([0,d]\) for all \(x \in [0,b]\). Then by the
inequality (1.3) one has:
\[ g_x \left( \frac{c + d}{2} \right) \leq \frac{1}{d - c} \int_c^d g_x(y) + mg_x\left( \frac{y}{m} \right) \, dy, \]
or
\[ f \left( x, \frac{c + d}{2} \right) \leq \frac{1}{d - c} \int_c^d f(x, y) + mf\left( \frac{x}{m} \right) \, dy \]
for all \( x \in [0, b] \). Integrating this inequality on \( [a, b] \), we have
\[ \frac{1}{b - a} \int_a^b f \left( x, \frac{c + d}{2} \right) \, dx \leq \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(x, y) + mf\left( \frac{x}{m} \right) \, dy \, dx. \]
By a similar argument applied to the mapping \( g_y : [0, b] \to \mathbb{R}, g_y(x) = f(x, y) \), we get
\[ \frac{1}{d - c} \int_c^d f \left( \frac{a + b}{2}, y \right) \, dy \leq \frac{1}{(d - c)(b - a)} \int_a^b \int_c^d f(x, y) + mf\left( \frac{x}{m} \right) \, dy \, dx. \]
Summing the inequalities (2.7) and (2.8), we get the inequality (2.6).

\[ \square \]

2.7. Remark. Choosing \( m = 1 \) in (2.6), we get the second inequality of (1.7).

3. Inequalities for co-ordinated \((\alpha, m)\)-convex functions

3.1. Definition. Consider the bidimensional interval \( \Delta := [0, b] \times [0, d] \) in \( [0, \infty)^2 \). The mapping \( f : \Delta \to \mathbb{R} \) is \((\alpha, m)\)-convex on \( \Delta \) if
\[ f(tx + (1 - t)z, ty + m(1 - t)w) \leq t^\alpha f(x, y) + m(1 - t^\alpha) f(z, w) \]
holds for all \((x, y), (z, w) \in \Delta \) and \((\alpha, m) \in [0, 1]^2 \), with \( t \in [0, 1] \).

A function \( f : \Delta \to \mathbb{R} \) which is \((\alpha, m)\)-convex on \( \Delta \) is called co-ordinated \((\alpha, m)\)-convex on \( \Delta \) if the partial mappings
\[ f_y : [0, b] \to \mathbb{R}, \quad f_y(u) = f(u, y) \]
and
\[ f_x : [0, d] \to \mathbb{R}, \quad f_x(v) = f(x, v) \]
are \((\alpha, m)\)-convex for all \( y \in [0, d] \) and \( x \in [0, b] \) with some fixed \((\alpha, m) \in [0, 1]^2 \).

Note that for \((\alpha, m) = (1, 1)\) and \((\alpha, m) = (1, m)\), one obtains the class of co-ordinated convex and of co-ordinated \(m\)-convex functions on \( \Delta \), respectively.

3.2. Lemma. Every \((\alpha, m)\)-convex mapping \( f : \Delta \to \mathbb{R} \) is \((\alpha, m)\)-convex on the co-ordinates, where \( \Delta = [0, b] \times [0, d] \) and \( \alpha, m \in [0, 1] \).

Proof. Suppose that \( f : \Delta \to \mathbb{R} \) is \((\alpha, m)\)-convex on \( \Delta \). Consider the function
\[ f_x : [0, d] \to \mathbb{R}, \quad f_x(v) = f(x, v), \quad (x \in [0, b]). \]
Then for $t \in [0,1]$, $(\alpha, m) \in [0,1]^2$ and $v_1, v_2 \in [0, d]$, one has

$$f_x (tv_1 + m (1 - t) v_2) = f (x, tv_1 + m (1 - t) v_2)$$

$$= f (tx + (1 - t) x, tv_1 + m (1 - t) v_2)$$

$$\leq t^\alpha f (x, v_1) + m (1 - t^\alpha) f (x, v_2)$$

$$= t^\alpha f_x (v_1) + m (1 - t^\alpha) f_x (v_2).$$

Therefore, $f_x (v) = f (x, v)$ is $(\alpha, m)$-convex on $[0, d]$. The fact that $f_y : [0, b] \to \mathbb{R}$, $f_y (u) = f (u, y)$ is also $(\alpha, m)$-convex on $[0, b]$ for all $y \in [0, d]$ goes likewise, and we shall omit the details. \hfill $\square$

3.3. Theorem. Suppose that $f : \Delta = [0, b] \times [0, d] \to \mathbb{R}$ is an $(\alpha, m)$-convex function on the co-ordinates on $\Delta$, where $(\alpha, m) \in (0, 1]^2$. If $0 \leq a < b \leq \infty$, $0 \leq c < d < \infty$ and $f_x \in L_1 [0, d]$, $f_y \in L_1 [0, b]$, then the following inequalities hold:

$$\frac{1}{b - a} \int_a^b f \left( x, \frac{c + d}{2} \right) dx + \frac{1}{d - c} \int_c^d f \left( \frac{a + b}{2}, y \right) dy$$

$$\leq \frac{1}{(d - c) (b - a)}$$

$$\times \int_c^d \int_a^b 2f (x, y) + m(2^\alpha - 1) \left( f \left( \frac{x, y}{m} \right) + f \left( \frac{x}{m}, y \right) \right) dx dy,$$

and

$$\frac{1}{(d - c) (b - a)} \int_c^d \int_a^b f (x, y) dx dy$$

$$\leq \frac{1}{2 (\alpha + 1) (b - a)} \min \{w_1, w_2\} + \frac{1}{2 (\alpha + 1) (d - c)} \min \{w_3, w_4\},$$

where

$$w_1 = \int_a^b f (x, c) dx + am \int_a^b f \left( x, \frac{d}{m} \right) dx$$

$$w_2 = \int_a^b f (x, d) dx + am \int_a^b f \left( x, \frac{c}{m} \right) dx$$

$$w_3 = \int_c^d f (a, y) dy + am \int_c^d f \left( \frac{b}{m}, y \right) dy$$

$$w_4 = \int_c^d f (b, y) dy + am \int_c^d f \left( \frac{a}{m}, y \right) dy.$$

Proof. Since $f : \Delta \to \mathbb{R}$ is co-ordinated $(\alpha, m)$-convex on $\Delta$ it follows that the mapping $g_x : [0, d] \to \mathbb{R}$, $g_x (y) = f (x, y)$ is $(\alpha, m)$-convex on $[0, d]$ for all $x \in [0, b]$. Then by the inequality (1.4) one has:

$$g_x \left( \frac{c + d}{2} \right) \leq \frac{1}{d - c} \int_c^d g_x (y) + m(2^\alpha - 1)g_x \left( \frac{y}{m} \right) dy,$$

that is

$$f \left( x, \frac{c + d}{2} \right) \leq \frac{1}{d - c} \int_c^d f (x, y) + m(2^\alpha - 1)f \left( x, \frac{y}{m} \right) dy,$$
where \(0 \leq c < d < \infty\) and \((\alpha, m) \in (0, 1]^2\). Integrating this inequality on \([a, b]\), we have

\[
\frac{1}{b-a} \int_a^b f \left( x, \frac{c+d}{2} \right) \, dx \leq \frac{1}{(d-c) (b-a)} \int_a^b \int_c^d f(x, y) + m(2^\alpha - 1) f \left( x, \frac{y}{m} \right) \, dy \, dx,
\]

(3.4)

where \(0 \leq a < b < \infty\).

By a similar argument applied for the mapping \(g_y : [0, b] \to [0, \infty)\), \(g_y(x) = f(x, y)\) with \(0 \leq a < b < \infty\), we get

\[
\frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) \, dy \leq \frac{1}{(d-c) (b-a)} \int_a^b \int_c^d f(x, y) + m(2^\alpha - 1) f \left( x, \frac{y}{m} \right) \, dy \, dx.
\]

(3.5)

Summing the inequalities (3.4) and (3.5), we get the inequality (3.2).

The inequality (3.3) can be obtained in a similar way to the proof of Theorem 2.3 by using (1.5). \(\square\)

3.4. Remark. If we take \(\alpha = 1\), (3.2) and (3.3) reduce to (2.6) and (2.1), respectively.

3.5. Theorem. Suppose that \(f : \Delta = [0, b] \times [0, d] \to \mathbb{R}\) is \((\alpha, m)\)-convex function on the co-ordinates on \(\Delta\), where \((\alpha, m) \in (0, 1]^2\). If \(0 \leq a < b < \infty\), \(0 \leq c < d < \infty\) and \(f_x \in L_1 [0, d]\), \(f_y \in L_1 [0, b]\), then the following inequality holds:

\[
\frac{1}{(b-a) (d-c)} \int_a^b \int_c^d f(x, y) \, dy \, dx \leq \frac{1}{4(\alpha + 1)} \left[ \frac{1}{b-a} \int_a^b f(x, c) \, dx + \frac{1}{b-a} \int_a^b f(x, d) \, dx \right.
\]

\[
+ \frac{m\alpha}{b-a} \int_a^b f \left( x, \frac{c}{m} \right) \, dx + \frac{m\alpha}{b-a} \int_a^b f \left( x, \frac{d}{m} \right) \, dx
\]

\[
+ \frac{1}{d-c} \int_c^d f(a, y) \, dy + \frac{1}{d-c} \int_c^d f(b, y) \, dy
\]

\[
+ \frac{m\alpha}{d-c} \int_c^d f \left( \frac{a}{m}, y \right) \, dy + \frac{m\alpha}{d-c} \int_c^d f \left( \frac{b}{m}, y \right) \, dy \right].
\]

(3.6)

Proof. Since \(f : \Delta \to \mathbb{R}\) is co-ordinated \((\alpha, m)\)-convex on \(\Delta\) it follows that the mapping \(g_x : [0, d] \to \mathbb{R}\), \(g_x(y) = f(x, y)\) is \((\alpha, m)\)-convex on \([0, d]\) for all \(x \in [0, b]\). Then by inequality (1.6) one has:

\[
\frac{1}{d-c} \int_c^d g_y(y) \, dy \leq \frac{1}{2} \left[ \frac{g_x(c) + g_x(d) + m\alpha (g_x \left( \frac{c}{m} \right) + g_x \left( \frac{d}{m} \right))}{\alpha + 1} \right],
\]

that is

\[
\frac{1}{d-c} \int_c^d f(x, y) \, dy \leq \frac{1}{2} \left[ \frac{f(x, c) + f(x, d) + m\alpha (f \left( x, \frac{c}{m} \right) + f \left( x, \frac{d}{m} \right))}{\alpha + 1} \right].
\]
where $0 \leq c < d < \infty$ and $(\alpha, m) \in (0, 1)^2$. Integrating this inequality on $[a, b]$, we have
\[
\frac{1}{(b - a) (d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx
\]
\[
\leq \frac{1}{2(\alpha + 1)} \left[ \frac{1}{b - a} \int_a^b f(x, c) \, dx + \frac{1}{d - c} \int_c^d f (x, d) \, dx \right.
\]
\[
+ \frac{\alpha}{b - a} \int_a^b f(x, c) \, dx + \frac{\alpha}{d - c} \int_c^d f (x, d) \, dx \]}
\]
where $0 \leq a < b < \infty$.

By a similar argument applied to the mapping $g_y : [0, b] \to [0, \infty)$, $g_y (x) = f(x, y)$ with $0 \leq a < b < \infty$, we get
\[
\frac{1}{(d - c) (b - a)} \int_c^d \int_a^b f(x, y) \, dx \, dy
\]
\[
\leq \frac{1}{2(\alpha + 1)} \left[ \frac{1}{d - c} \int_c^d f (a, y) \, dy + \frac{1}{b - a} \int_a^b f (b, y) \, dy \right.
\]
\[
+ \frac{\alpha}{d - c} \int_c^d f (a, y) \, dy + \frac{\alpha}{b - a} \int_a^b f (b, y) \, dy \]}
\]

Summing the inequalities (3.7) and (3.8), we get the inequality (3.6). \(\square\)

3.6. Corollary. Choosing $m = 1$ in Theorem 3.5, we get the following inequality
\[
\frac{1}{(b - a) (d - c)} \int_a^b \int_c^d f(x, y) \, dy \, dx
\]
\[
\leq \frac{1}{4(\alpha + 1)} \left[ \frac{1}{b - a} \int_a^b f(x, c) \, dx + \frac{1}{d - c} \int_c^d f (x, d) \, dx \right.
\]
\[
+ \frac{\alpha}{b - a} \int_a^b f(x, c) \, dx + \frac{\alpha}{d - c} \int_c^d f (x, d) \, dx \]}
\]
\[\]
\[
+ \frac{1}{d - c} \int_c^d f (a, y) \, dy + \frac{1}{b - a} \int_a^b f (b, y) \, dy
\]
\[
\left. + \frac{\alpha}{d - c} \int_c^d f (a, y) \, dy + \frac{\alpha}{b - a} \int_a^b f (b, y) \, dy \right].
\]

3.7. Remark. Choosing $(\alpha, m) = (1, 1)$ in (3.6), we get the third inequality of (1.7).

References

