IMPROVED BOUNDS FOR THE
SPECTRAL RADIUS OF DIGRAPHS

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Abstract

Let $G = (V, E)$ be a digraph with $n$ vertices and $m$ arcs without loops and multi-arcs. The spectral radius $\rho(G)$ of $G$ is the largest eigenvalue of its adjacency matrix. In this note, we obtain two sharp upper and lower bounds on $\rho(G)$. These bounds improve those obtained by G. H. Xu and C.-Q Xu (Sharp bounds for the spectral radius of digraphs, Linear Algebra Appl. 430, 1607–1612, 2009).

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1. Introduction

Let $G$ be a digraph with $n$ vertices and $m$ arcs without loops and multi-arcs on the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. If $(u, v)$ be an arc of $G$, then $u$ is called the initial vertex and $v$ the terminal vertex of this arc. The outdegree $d^+_i$ of a vertex $v_i$ in the digraph $G$ is defined to be the number of arcs in $G$ with initial vertex $v_i$. Let $d^+_1, d^+_2, \ldots, d^+_n$ be the outdegree sequence and $\delta^+(G)$ the minimum outdegree of $G$. For convenience, we sometimes abbreviate $\delta^+(G)$ to $\delta^+$.

Let $t^+_i$ be the sum of the outdegrees of all vertices in $N^+_i(v_i) = \{v_j : (v_i, v_j) \in E\}$, and call it the 2-outdegree. Moreover, call $m^+_i = \frac{t^+_i}{d^+_i}$ the average 2-outdegree, $1 \leq i \leq n$. If the average 2-outdegrees of the vertices in $V$ are the same, we call $G$ an average 2-outdegree regular digraph. If $V = U \cup W$, and the average 2-outdegrees of the vertices in $U$ and

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W are $m_1^+$ and $m_2^+$, respectively, we call G an average 2-outdegree semiregular digraph. Now we define:

$$\left(\alpha_{m+}\right)_i = \sum_{(v_i,v_j) \in E} (d_i^+) \alpha \quad \text{and} \quad \left(\alpha_{m+}\right)_i = \frac{\sum_{(v_i,v_j) \in E} (d_i^+) \alpha}{(d_i^+)^\alpha},$$

where $\alpha$ is a real number. Note that $d_i^+ = (0^+) = (0^+) = (1^+) = (1^+) = m_1^+$.

The spectral radius $\rho(G)$ of G is defined to be largest eigenvalue of its adjacency matrix $A(G)$. Recently, the spectral radius of a digraph has been well studied in [2,3,5,6].

In this note, we present two sharp upper and lower bounds on the spectral radius of a digraph $G$, and obtain some known results from it. In fact, for undirected graphs, the following result has been obtained in [4].

1.1. Lemma. [4] Let $G$ be a connected undirected graph. Then

$$\rho(G) \leq \min_\alpha \max_{(v_i,v_j) \in E} \left\{ \sqrt{\alpha_{m+} \alpha_{m+}} \right\},$$

where $d_i$ is the degree of $v_i$ and $(\alpha_{m+})_i = \frac{\sum_{(v_i,v_j) \in E} (d_i^+) \alpha}{(d_i^+)^\alpha}$. Moreover, the equality holds for a particular value of $\alpha$ if and only if $(\alpha_{m+})_1 = (\alpha_{m+})_2 = \cdots = (\alpha_{m+})_n$, or $G$ is a bipartite graph with the partition $\{v_1,\ldots,v_{n_1}\} \cup \{v_{n_1+1},\ldots,v_n\}$ and $(\alpha_{m+})_1 = \cdots = (\alpha_{m+})_{n_1}$.

$(\alpha_{m+})_{n_1+1} = \cdots = (\alpha_{m+})_n$. \hfill $\Box$

Now, we will give a generation of this result on the spectral radius for digraphs.

2. Upper bound on the spectral radius of digraphs

Throughout this section, let $G$ be a digraph with $n$ vertices and $m$ arcs without loops and multi-arcs. Let $(d_1^+, d_2^+, \ldots, d_n^+)$ be the outdegree sequence and $A(G)$ the adjacency matrix of $G$. Let

$$D = \text{diag} ((d_1^+)^\alpha, \ldots, (d_n^+)^\alpha).$$

2.1. Lemma. [1] Let $A$ be a nonnegative matrix of order $n$. Let $R_i$ be the sum of the $i$th row of $A$. Then

$$\min \{R_i : 1 \leq i \leq n\} \leq \rho(A) \leq \max \{R_i : 1 \leq i \leq n\}.$$

If $A$ is irreducible, then equality holds in both cases if and only if $R_1 = R_2 = \cdots = R_n$. \hfill $\Box$

Now, we give our main result of this section.

2.2. Theorem. Let $G$ be a digraph with $n$ vertices, and $\delta^+$ the minimum outdegree of $G$, $\delta^+ \geq 1$. Then

$$\rho(G) \leq \min_\alpha \max_{(v_i,v_j) \in E} \left\{ \sqrt{\alpha_{m+} \alpha_{m+}} \right\}. \quad (1)$$

Moreover, if $G$ is a strongly connected digraph, equality holds for a particular value of $\alpha$ if and only if $(\alpha_{m+})_1 = (\alpha_{m+})_2 = \cdots = (\alpha_{m+})_{n_1}$, or $G$ is a bipartite graph with the partition $\{v_1,\ldots,v_{n_1}\} \cup \{v_{n_1+1},\ldots,v_n\}$ and $(\alpha_{m+})_1 = \cdots = (\alpha_{m+})_{n_1}$, $(\alpha_{m+})_{n_1+1} = \cdots = (\alpha_{m+})_n$. \hfill $\Box$
Proof. Note that \( \rho(G) = \rho(\bar{D}^{-1}A(G)\bar{D}) \). Now the \((i,j)\)th element of \( \bar{D}^{-1}A(G)\bar{D} \) is
\[
\begin{cases}
  \left(\frac{d_i^+}{d_j^+}\right)^\alpha & \text{if } (v_i, v_j) \in E, \\
  0 & \text{otherwise}.
\end{cases}
\]

Let \( X = (x_1, x_2, \ldots, x_n)^T \) be an eigenvector of \( \bar{D}^{-1}A(G)\bar{D} \) corresponding to the eigenvalue \( \rho(G) \). We can assume that one eigen-component, say \( x_i \), is equal to 1 and the other eigen-components are less than or equal to 1, that is, \( x_i = 1 \) and \( 0 < x_k \leq 1 \), for all \( k \).

Let \( x_j = \max \{ x_k : (v_i, v_k) \in E \} \).

Since
\[
\bar{D}^{-1}A(G)\bar{D}X = \rho(G)X,
\]
we have
\[
\rho(G)x_i = \sum_k \left\{ \left(\frac{d_k^+}{d_i^+}\right)^\alpha x_k : (v_i, v_k) \in E \right\} \leq (\alpha_m^+) x_j,
\]
(2)
\[
\rho(G)x_j = \sum_k \left\{ \left(\frac{d_k^+}{d_j^+}\right)^\alpha x_k : (v_j, v_k) \in E \right\} \leq (\alpha_m^+) j,
\]
(3)

From (2) and (3), we get
\[
\rho(G) \leq \sqrt{(\alpha_m^+) (\alpha_m^+)}.
\]

Now we assume that in (1) equality holds for a particular value of \( \alpha \). Then all the inequalities in the above argument must be equalities. In particular, we have from (2) that \( x_k = x_j \) for all \( k \) such that \( (v_i, v_k) \in E \). Also, from (3) we have that \( x_k = x_i = 1 \) for all \( k \) such that \( (v_j, v_k) \in E \). Let \( U = \{ v_k \in V(G) : x_k = 1 \} \). Then \( v_i \in U \).

If \( x_j = 1 \), then we will show that \( U = V(G) \). Otherwise, if \( U \neq V(G) \), there exist vertices \( v_a, v_b \in U, v_c \notin U \), such that \( (v_a, v_b) \in E \) and \( (v_b, v_c) \in E \) since \( G \) is strongly connected. Therefore, from
\[
\rho(G)x_a = \sum_k \left\{ \left(\frac{d_k^+}{d_a^+}\right)^\alpha x_k : (v_a, v_k) \in E \right\} \leq \alpha_m^+ \alpha_a
\]
and
\[
\rho(G)x_b = \sum_k \left\{ \left(\frac{d_k^+}{d_b^+}\right)^\alpha x_k : (v_b, v_k) \in E \right\} < \alpha_m^+ \alpha_b,
\]
we have
\[
\rho(G) < \sqrt{\alpha_m^+ \alpha_a (\alpha_m^+) \alpha_b},
\]
which contradicts that equality holds in (1). Thus \( U = V(G) \) and
\[
\alpha_m^+ \alpha_a = \alpha_m^+ \alpha_b = \cdots = \alpha_m^+ \alpha_n = \rho(G).
\]

Suppose that \( x_j < 1 \), and let \( W = \{ v_k \in V(G) : x_k = x_j \} \). Then, \( N_G(v_j) \subseteq U \) and \( N_G(v_j) \subseteq W \). Now we show that \( N_G(N_G(v_j)) \subseteq U \). Let \( v_r \in N_G(N_G(v_j)) \), there exists a vertex \( v_p \) such that \( (v_r, v_p) \in E \) and \( (v_r, v_p) \in E \). Therefore,
\[
x_p = x_j \text{ and } \rho(G)x_p = \sum_k \left\{ \left(\frac{d_k^+}{d_p^+}\right)^\alpha x_k : (v_p, v_k) \in E \right\} \leq (\alpha_m^+) x_p.
\]
Moreover, if $G$ be a digraph with $\rho(G)$, then we get
\[ \rho(G)^2 = (\alpha_{m+})_1 (\alpha_{m+})_p, \]
where $\rho(G)$ is the spectral radius of $G$. We have
\[ \rho(G)^2 \leq (\alpha_{m+})_1 (\alpha_{m+})_p, \]
which shows that $x_v = 1$. Hence $N_G (N_G (v_i)) \subseteq U$. By a similar argument, we can show that $N_G (N_G (v_j)) \subseteq W$. Continuing the procedure, since $G$ is strongly connected it is easy to see that $V = U \cup W$, and that the directed subgraphs induced by $U$ and $W$, respectively, are empty digraphs. Hence $G$ is bipartite. Moreover, $(\alpha_{m+})_p$ are the same for all $v_p \in U$ and $(\alpha_{m+})_p$ are the same for all $v_p \in W$.

Conversely, if $G$ is a graph with $(\alpha_{m+})_1 = (\alpha_{m+})_2 = \cdots = (\alpha_{m+})_n$, then the equality in (1) is satisfied. Let $G$ be a bipartite graph with bipartition $V = U \cup W$ and $(\alpha_{m+})_i = a$ for $v_i \in U$. $(\alpha_{m+})_i = b$ for $v_i \in W$. Let $M = K^{-1} \left( \bar{D}^{-1} A(G) \bar{D} \right) K$, where $K = \text{diag} \{ \sqrt{(\alpha_{m+})_1}, \ldots, \sqrt{(\alpha_{m+})_n} \}$.

Note that the $(i,j)$th element of $M$ is
\[
\begin{cases} 
\sqrt{\frac{d_i^+}{d_j^+}} & \text{if } (v_i, v_j) \in E \text{ and } v_i \in U, \\
\sqrt{\frac{d_i^+}{d_j^+}} & \text{if } (v_i, v_j) \in E \text{ and } v_i \in W, \\
0 & \text{otherwise}.
\end{cases}
\]
So each row sum of the matrix $M$ is equal to $\sqrt{ab}$. Thus, by Lemma 2.1, we have $\rho(G) = \rho(M) = \sqrt{ab}$.

2.3. Corollary. Let $G$ be a graph with $n$ vertices and let $\delta^+$ be the minimum outdegree of $G$, $\delta^+ \geq 1$. Then
\[ \rho(G) \leq \min_{1 \leq i \leq n} \left\{ (\alpha_{m+})_i \right\}. \]
Moreover, if $G$ is a strongly connected digraph, equality holds for a particular value of $\alpha$ if and only if $\rho(G)^2 = (\alpha_{m+})_1 (\alpha_{m+})_n$.

If $\alpha = 1$ in (1), then we get the following result.

2.4. Corollary. [5] Let $G$ be a digraph on $n$ vertices and $\delta^+$ the minimum outdegree of $G$, $\delta^+ \geq 1$. Then
\[ \rho(G) \leq \max \left\{ \sqrt{m_i^+ m_j^+} : (v_i, v_j) \in E \right\}. \]
Moreover, if $G$ is a strongly connected digraph, equality holds if and only if $G$ is average 2-outdegree regular or average 2-outdegree semiregular.

3. Lower bound on the spectral radius of digraphs

3.1. Theorem. Let $G$ be a digraph with $n$ vertices and let $\delta^+$ be the minimum outdegree of $G$, $\delta^+ \geq 1$. Then
\[ \rho(G) \geq \max_{\alpha} \min_{\{v_i, v_j\} \in E} \left\{ \sqrt{(\alpha_{m+})_1 (\alpha_{m+})_j} \right\}. \]
Moreover, if $G$ is a strongly connected digraph, equality holds for a particular value of $\alpha$ if and only if $\rho(G)^2 = (\alpha_{m+})_1 (\alpha_{m+})_n$, or $G$ is a bipartite graph with the partition $\{v_1, \ldots, v_{n1}\} \cup \{v_{n1+1}, \ldots, v_n\}$ and $\alpha_{m+})_1 = \cdots = (\alpha_{m+})_{n1}, \alpha_{m+})_{n1+1} = \cdots = (\alpha_{m+})_n$. 

\[ \]
Proof. Let $X = (x_1, x_2, \ldots, x_n)^T$ be an eigenvector of $\bar{D}^{-1}A(G)\bar{D}$ corresponding to the eigenvalue $\rho(G)$. We can assume that one eigen-component, say $x_i$, is equal to 1 and the other eigen-components are greater than or equal to 1, that is, $x_i = 1$ and $x_k \geq 1$ for all $k \neq i$. Let $x_j = \min \{x_k : (v_i, v_k) \in E\}$.

Since

$$\bar{D}^{-1}A(G)\bar{D}X = \rho(G)X,$$

we have

$$\rho(G)x_i = \sum_k \left\{ \frac{(d^+_k)^\alpha}{(d^+_i)^\alpha} x_k : (v_i, v_k) \in E \right\} \geq (\alpha_{m^+_+})_i x_j,$$

(7)

$$\rho(G)x_j = \sum_k \left\{ \frac{(d^+_k)^\alpha}{(d^+_j)^\alpha} x_k : (v_j, v_k) \in E \right\} \geq (\alpha_{m^+_+})_j.$$

(8)

From (7) and (8), we get

$$\rho(G) \geq \sqrt{(\alpha_{m^+_+})_i (\alpha_{m^+_+})_j}.$$

Similarly as in the proof of the Theorem 2.2, we can show that equality holds for a particular value of $\alpha$ if and only if $(\alpha_{m^+_+})_1 = (\alpha_{m^+_+})_2 = \cdots = (\alpha_{m^+_+})_n$, or $G$ is a bipartite graph with the partition $\{v_1, \ldots, v_{n_1}\} \cup \{v_{n_1+1}, \ldots, v_n\}$ and $(\alpha_{m^+_+})_1 = \cdots = (\alpha_{m^+_+})_{n_1}$, $(\alpha_{m^+_+})_{n_1+1} = \cdots = (\alpha_{m^+_+})_n$.

3.2. Corollary. Let $G$ be a digraph with $n$ vertices and let $\delta^+$ be the minimum outdegree of $G$, $\delta^+ \geq 1$. Then

$$\rho(G) \geq \max_{1 \leq i \leq n} \{(\alpha_{m^+_+})_i\}.$$

Moreover, if $G$ is a strongly connected digraph, equality holds for a particular value of $\alpha$ if and only if $(\alpha_{m^+_+})_1 = (\alpha_{m^+_+})_2 = \cdots = (\alpha_{m^+_+})_n$. □

If $\alpha = 1$ in (6), the we get the following result.

3.3. Corollary. [5] Let $G$ be a strongly connected digraph. Then

$$\rho(G) \geq \min \left\{ \sqrt{m^+_+ m^-_+} : (v_i, v_j) \in E \right\}.$$

Moreover, equality holds if and only if $G$ is average 2-outdegree regular or average 2-outdegree semiregular. □

3.4. Example. Let $G$ be a digraph with adjacency matrix

$$\begin{bmatrix}
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}. $$

Then the bound (1) is 1.847 when $\alpha = 0.5$, and the bound (5) from [5] is 2. For the same graph, the bound (6) is 1.414 when $\alpha = 0.5$, and the bound (10) from [5] is 1.154. Thus in both cases, the results obtained in this paper for $\alpha = 0.5$ are better than the bounds obtained in [5].

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References


