Generalized $\omega\alpha$-Closed Sets in Topological Spaces

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Abstract - The aim of this paper is to introduce a new class of closed sets called $g\omega\alpha$-closed sets using $\omega\alpha$-closed sets in topological spaces. This class is independent of $\omega\alpha$-closed sets. This new class of set lies between the class of $\alpha$-closed sets and the class of $\alpha g$-closed sets. Some of their properties are investigated. We also define and study the $g\omega\alpha$-closure and $g\omega\alpha$-interior in topological spaces.

Keywords - Topological spaces, generalized closed sets, $\omega\alpha$-closed sets, $g\omega\alpha$-closed sets and $g\omega\alpha$-open sets.

1 Introduction

In 1969 Levine [9] gives the concept and properties of generalized closed (briefly g-closed) sets and the complement of g-closed set is said to be g-open set. In 1982 Mashhour et.al [13] introduced and studied the concept of pre-open set. Later Maki et.al [12], Dontechev [6], Gyanambal [7], Arya and Nour [3] and Bhattacharya and Lahiri [4] introduced and studied the concepts of gp-closed, gsp-closed, gpr-closed, gs-closed, sg-closed and $\alpha g$-closed and their compliments are respective open sets.


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2 Preliminaries

Throughout this paper space \((X, \tau)\) and \((Y, \sigma)\) (or simply \(X\) and \(Y\)) always denote topological space on which no separation axioms are assumed unless explicitly stated. For a subset \(A\) of a space \((X, \tau)\) \(\text{Cl}(A)\), \(\text{Int}(A)\) and \(A^c\) denote the Closure of \(A\), Interior of \(A\) and Compliment of \(A\) respectively.

**Definition 2.1.** A subset \(A\) of a topological space \((X, \tau)\) is called,

(i) **Semi-open set** [8] if \(A \subseteq \text{Cl}(\text{Int}(A))\) and **Semi-closed set** if \(\text{Int}(\text{Cl}(A)) \subseteq A\).

(ii) **Pre-open set** [13] if \(A \subseteq \text{Int}(\text{Cl}(A))\) and **Pre-closed set** if \(\text{Cl}(\text{Int}(A)) \subseteq A\).

(iii) **\(\alpha\)-open set** [16] if \(A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))\) and **\(\alpha\)-closed set** if \(\text{Cl}(\text{Int}(\text{Cl}(A))) \subseteq A\).

(iv) **Semi-pre-open set** [2] \((=\beta\)-open set [1]\) if \(A \subseteq \text{Cl}(\text{Int}(\text{Cl}(A)))\) and **semi-pre-closed** \((=\beta\)-closed set [1]\) if \(\text{Cl}(\text{Int}(\text{Cl}(A))) \subseteq A\).

(v) **Regular-open** [7] if \(A = \text{Int}(\text{Cl}(A))\) and **Regular-closed** if \(A = \text{Cl}(\text{Int}(A))\).

The \(\alpha\)-closure of \(A\) is the smallest \(\alpha\)-closed set containing \(A\), and this is denoted by \(\alpha\text{Cl}(A)\). Similarly the semi-closure (resp pre-closure and semi-pre-closure) of a set \(A\) in a topological space \((X, \tau)\) is the intersection of all semi-closed (resp pre-closed and semi-pre-closed) sets containing \(A\) and is denoted by \(\text{scl}(A)\) (resp \(\text{pcl}(A)\) and \(\text{spcl}(A)\)).

**Definition 2.2.** A subset of a topological space \((X, \tau)\) is called a,

(i) **Generalized closed** (briefly **g-closed**) set [9] if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).

(ii) **Semi-generalized closed** (briefly **sg-closed**) set [4] if \(\text{scl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is Semi-open in \(X\).

(iii) **Generalized semi-closed** (briefly **gs-closed**) set [3] if \(\text{scl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).

(iv) **Generalized \(\alpha\)-closed** (briefly **g\(\alpha\)-closed**) set [10] if \(\alpha\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(\alpha\)-open in \(X\).

(v) **\(\alpha\)-generalized closed** (briefly **\(\alpha\)g-closed**) set [11] if \(\alpha\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).

(vi) **Generalized pre-closed** (briefly **gp-closed**) set [12] if \(\text{pcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).

(vii) **Generalized semi-pre-closed** (briefly **gsp-closed**) set [6] if \(\text{spcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).

(viii) **Generalized pre-regular-closed** (briefly **gpr-closed**) set [7] if \(\text{pcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular-open in \(X\).

(ix) **Weakly closed** (briefly **\(\omega\)-closed**) set [21] if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is semi-open in \(X\).

(x) **Weakly generalized closed** (briefly **\(\omega\)g-closed**) set [20] if \(\text{cl}(\text{int}(A)) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is open in \(X\).

(xi) **Strongly generalized closed** (briefly **g\(^*\)-closed**) set [18] if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(g\)-open in \(X\).

(xii) **Regular generalized closed** (briefly **rg-closed**) set [17] if \(\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular-open in \(X\).

(xiii) **\(\alpha\)-generalized regular closed** (briefly **\(\alpha\)gr-closed**) set [23] if \(\alpha\text{cl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is regular-open in \(X\).

(xiv) **\(g^*\)-preclosed** (briefly **\(g^*\)p-closed**) [22] if \(\text{pcl}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is \(g\)-open
in X.

(xiv) \( \omega \alpha \) closed set [5] if \( \alpha \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \omega \)-open in \( X \).

The compliment of the above mentioned closed sets are their open sets respectively.

\section*{3 \( g\omega \alpha \)-closed sets in Topological spaces.}

In this section we introduce \( g\omega \alpha \)-closed sets in topological space and study some of their properties.

\textbf{Definition 3.1.} A subset \( A \) of a topological space \((X, \tau)\) is called a generalized \( \omega \alpha \)-closed (\( g\omega \alpha \)-closed) set if \( \alpha \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \omega \alpha \)-open in \( X \).

\textbf{Theorem 3.2.} Every closed set in \( X \) is \( g\omega \alpha \)-closed set.

\textbf{Proof:} Let \( A \) be a closed set in a topological space \( X \), let \( G \) be any \( \omega \alpha \)-open sets in \( X \) such that \( A \subseteq G \). Since \( A \) is closed, we have \( \text{cl}(A) = A \), but \( \alpha \text{cl}(A) \subseteq \text{cl}(A) \) is always true. So \( \alpha \text{cl}(A) \subseteq \text{cl}(A) \subseteq G \). Therefore \( \alpha \text{cl}(A) \subseteq G \). Hence \( A \) is \( g\omega \alpha \)-closed set.

The converse of the above theorem need not be true as seen from the following example.

\textbf{Example 3.3.} Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{a, b\}\} \) then the set \( A = \{a, c\} \) is \( g\omega \alpha \)-closed but not closed.

\textbf{Theorem 3.4.} Every \( \alpha \)-closed set in \( X \) is \( g\omega \alpha \)-closed set.

\textbf{Proof:} Let \( A \) be a \( \alpha \)-closed set in a topological space \( X \), let \( U \) be any \( \omega \alpha \)-open set in \( X \) such that \( A \subseteq U \). Since \( A \) is \( \alpha \)-closed we have \( \alpha \text{cl}(A) = A \subseteq U \). Therefore \( \alpha \text{cl}(A) \subseteq U \). Hence \( A \) is \( g\omega \alpha \)-closed set.

The converse of the above theorem need not be true as seen from the following example.

\textbf{Example 3.5.} Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}, \{b, c\}\} \) then the set \( A = \{b\} \) is \( g\omega \alpha \)-closed but not \( \alpha \)-closed in \( X \).

\textbf{Theorem 3.6.} Every \( g\omega \alpha \)-closed set in \( X \) is \( \alpha g \)-closed set in \( X \).

\textbf{Proof:} Let \( A \) be \( g\omega \alpha \)-closed set in \( X \). Let \( U \) be any open set in \( X \), such that \( A \subseteq U \). Since every open set is \( \omega \alpha \)-open set and \( A \) is \( g\omega \alpha \)-closed, we have \( \alpha \text{cl}(A) \subseteq U \) and hence \( A \) is \( \alpha g \)-closed set in \( X \).

The converse of the above theorem need not be true as seen from the following example.

\textbf{Example 3.7.} Let \( X = \{a, b, c\} \) and \( \tau = \{X, \phi, \{a\}\} \) then the set \( A = \{a, b\} \) is \( \alpha g \)-closed but not \( g\omega \alpha \)-closed in \( X \).
Remark 3.8. From the theorem 3.4 and 3.6 it follows that $g\omega\alpha$-closed set properly lies between $\alpha$-closed set and $\alpha g$-closed set.

Theorem 3.9. Every regular-closed (resp $\omega$-closed, $g\alpha$-closed) set is $g\omega\alpha$-closed set.

Proof: The proof is obvious from theorem 3.2.

The converse of the above theorem need not be true as seen from the following example.

Example 3.10. In Example 3.3 the set $A = \{a, c\}$ is $g\omega\alpha$-closed but not regular-closed ($\omega$-closed, $ga$-closed) set in $X$.

Theorem 3.11. Every $g\omega\alpha$-closed set in $X$ is $gs$-closed (resp $gp$-closed, $gsp$-closed, $gpr$-closed, $rg$-closed, $\omega g$-closed, $\alpha gr$-closed, $g^*p$-closed) set in $X$.

Proof: Since every open set is $\omega\alpha$-open [5], the proof follows.

The converse of the above theorem need not be true as seen from the following example.

Example 3.12. In Example 3.7, the set $A = \{a, b\}$ is $gs$-closed (gp-closed, gsp-closed, gpr-closed, rg-closed, $\omega g$-closed, $\alpha gr$-closed) but not $g\omega\alpha$-closed in $X$.

Remark 3.14. The concept of $g\omega\alpha$-closed set is independent of the concept of sets namely $p$-closed, $sp$-closed, semi-closed, $g$-closed, $sg$-closed, $g^*$-closed, $g^*s$-closed, $\omega\alpha$-closed sets as seen from the following example.

Example 3.15. In Example 3.10, the set $A = \{a, c\}$ is $g\omega\alpha$-closed but not $p$-closed, $sp$-closed, semi-closed, $sg$-closed, $g^*$-closed, and the set $B = \{b\}$ is $g\omega\alpha$-closed but not $g$-closed and $g^*$-closed in $X$.

Example 3.16. In Example 3.5, the set $A = \{b\}$ is $g\omega\alpha$-closed but not $\omega\alpha$-closed set in $X$.

Example 3.17. Let $X = \{a, b, c, d\}$ and $\tau = \{X, \phi, \{a\}, \{a, c\}\}$ then the set $A = \{a, b, c\}$ is $g\omega\alpha$-closed and $sp$-closed set in $X$.

Example 3.18. Let $X = \{a, b, c\}$ and $\tau = \{X, \phi, \{a\}, \{a, b\}\}$ then the set $A = \{a, b\}$ is semi-closed, $g^*$-closed and $g^*s$-closed but not $g\omega\alpha$-closed set in $X$.

Example 3.19. In Example 3.13, the set $A = \{a, b\}$ is $g$-closed, $g^*$-closed, and $\omega\alpha$-closed but not $g\omega\alpha$-closed set in $X$.

Theorem 3.20. Union of two $g\omega\alpha$-closed sets are $g\omega\alpha$-closed set.

Proof: Let $A$ and $B$ be two $g\omega\alpha$-closed sets in $(X, \tau)$, let $G$ be any $\omega\alpha$-open set in $(X, \tau)$, such that $A \cup B \subseteq G$. Then $A \subseteq G$ and $B \subseteq G$. Since $A$ and $B$ are $g\omega\alpha$-closed sets, $\alpha cl(A) \subseteq G$ and $\alpha cl(B) \subseteq G$. Therefore $\alpha cl(A) \cup \alpha cl(B) = \alpha cl(A \cup B) \subseteq G$. Hence $A \cup B$ is $g\omega\alpha$-closed set.
Theorem 3.21. If a subset $A$ of $X$ is $g\omega\alpha$-closed in $(X, \tau)$ then $\alpha cl(A)-A$ does not contain any non empty $\omega\alpha$-closed set in $(X, \tau)$.

Proof: Suppose $A$ is $g\omega\alpha$-closed and $F$ be a non empty $\omega\alpha$-closed subset of $\alpha \text{cl}(A)-A$. Then $F \subseteq \alpha \text{cl}(A) \cap (X-A)$. Since $(X-A)$ is $\omega\alpha$-open and $A$ is $g\omega\alpha$-closed, $\alpha \text{cl}(A) \subseteq (X-A)$, therefore $F \subseteq (X-\alpha \text{cl}(A))$. Thus $F \subseteq \alpha \text{cl}(A) \cap (X-\alpha \text{cl}(A)) = \phi$. That is $F = \phi$. Thus $\alpha \text{cl}(A)-A$ does not contain any non-empty $\omega\alpha$-closed set in $(X, \tau)$.

However the converse of the above theorem need not be true as seen from the following example.

Example 3.22. In Example 3.17, the set $A = \{a, b\}$ then $\alpha \text{cl}(A)-A = \{c, d\}$ does not contain non empty $\omega\alpha$-closed set. But $A$ is not $g\omega\alpha$-closed set in $(X, \tau)$.

Theorem 3.23. If $A$ is $g\omega\alpha$-closed set in $X$ and $A \subseteq B \subseteq \alpha \text{cl}(A)$ then $B$ is also $g\omega\alpha$-closed set in $X$.

Proof: It is given that $A$ is $g\omega\alpha$-closed set in $X$. To prove $B$ is also $g\omega\alpha$-closed set of $X$. Let $U$ be an $\omega\alpha$-open set of $X$, such that $B \subseteq U$. Since $A \subseteq B$, we have $A \subseteq U$. Since $A$ is $g\omega\alpha$-closed, and $\alpha \text{cl}(A) \subseteq U$. Now $\alpha \text{cl}(B) \subseteq \alpha \text{cl}(\alpha \text{cl}(A)) = \alpha \text{cl}(A) \subseteq U$. So $\alpha \text{cl}(B) \subseteq U$. Hence $B$ is $g\omega\alpha$-closed set in $X$.

However the converse of the above theorem need not be true as seen from the following example.

Example 3.24. In Example 3.5, the set $A = \{a\}$ and $B = \{a, b\}$ such that $A$ and $B$ are $g\omega\alpha$-closed sets but $A \subseteq B \not\subseteq \alpha \text{cl}(A)$.

Theorem 3.25. For each $x \in X$ either $x$ is $\omega\alpha$-closed or $x^c$ is $g\omega\alpha$-closed in $X$.

Proof: Suppose $\{x\}$ is not $\omega\alpha$-closed in $X$, then $\{x^c\}$ is not $\omega\alpha$-open and the only $\omega\alpha$-open set containing $\{x^c\}$ is the space $X$ itself. Therefore $\alpha \text{cl}(\{x^c\}) \subseteq X$ and hence $\{x^c\}$ is $g\omega\alpha$-closed set in $(X, \tau)$.

Theorem 3.26. Let $A$ be $g\omega\alpha$-closed in $(X, \tau)$. Then $A$ is $\alpha$-closed if and only if $\alpha \text{cl}(A)-A$ is $\omega\alpha$-closed.

Proof: Necessity: Suppose $A$ be $\alpha$-closed. Then $\alpha \text{cl}(A) = A$ and so $\alpha \text{cl}(A)-A = \phi$, which is $\omega\alpha$-closed.
Sufficiency: Suppose $\alpha \text{cl}(A)-A$ is $\omega\alpha$-closed. Then $\alpha \text{cl}(A)-A = \phi$, since $A$ is $g\omega\alpha$-closed. That is $\alpha \text{cl}(A)-A$ or $A$ is $\alpha$-closed.

Theorem 3.27. Let $A \subseteq Y \subseteq X$, and suppose that $A$ is $g\omega\alpha$-closed set in $X$. Then $A$ is $g\omega\alpha$-closed relative to $Y$.

Proof: Let $A \subseteq Y \cap G$ where $G$ is $\omega\alpha$-open. Then $A \subseteq G$ and hence $\alpha \text{cl}(A) \subseteq G$. This implies that $Y \cap \alpha \text{cl}(A) \subseteq Y \cap G$. Thus $A$ is $g\omega\alpha$-closed relative to $Y$.

Now we introduce the following.
Definition 3.28. A subset $A$ of a topological space $(X, \tau)$ is called $g\omega\alpha$-open set if its compliment $A^c$ is $g\omega\alpha$-closed.

Theorem 3.29. A subset $A$ of $(X, \tau)$ is $g\omega\alpha$-open set if and only if $U \subseteq \alpha \text{int}(A)$ whenever $U$ is $\omega\alpha$-closed and $U \subseteq A$.

Proof: Assume that $A$ is $g\omega\alpha$-open in $X$ and $U$ is $\omega\alpha$-closed set of $(X, \tau)$ such that $U \subseteq A$. Then $X-A$ is a $g\omega\alpha$-closed set in $(X, \tau)$. Also $X-A \subseteq X-U$ and $X-U$ is $\omega\alpha$-open set of $(X, \tau)$. This implies that $\alpha\text{cl}(X-A) \subseteq X-U$. But $\alpha\text{cl}(X-A) = X-\alpha\text{int}(A)$. Thus $X-\alpha\text{int}(A) \subseteq X-U$. So $U \subseteq \alpha\text{int}(A)$.

Conversely: Suppose $U \subseteq \alpha\text{int}(A)$ whenever $U$ is $\omega\alpha$-closed and $U \subseteq A$. To prove that $A$ is $g\omega\alpha$-open. Let $G$ be $\omega\alpha$-open set of $(X, \tau)$ such that $X-A \subseteq G$. Then $X-G \subseteq \alpha\text{int}(A)$. So $X-G \subseteq \alpha\text{int}(A)$, $X-\alpha\text{int}(A) \subseteq G$. But $\alpha\text{cl}(X-A) = X-\alpha\text{int}(A)$. Thus $\alpha\text{cl}(X-A) \subseteq G$. That is $X-A$ is $g\omega\alpha$-closed set and hence $A$ is $g\omega\alpha$-open.

Theorem 3.30. If $A$ is $\omega\alpha$-open and $g\omega\alpha$-closed set then $A$ is $\alpha$-closed.

Proof: Since $A \subseteq A$ and $A$ is $\omega\alpha$-open and $g\omega\alpha$-closed, we have $\alpha\text{cl}(A) \subseteq A$. Thus $\alpha\text{cl}(A) = A$. Hence $A$ is $\alpha$-closed set of $(X, \tau)$.

Theorem 3.31. A regular open $g\omega\alpha$-closed set is preclosed and hence clopen.

Proof: Let $A$ be regular open $g\omega\alpha$-closed. Since regular open set is $\omega\alpha$-open, $\alpha\text{cl}(A) \subseteq A$. This implies $A$ is $\alpha$-closed. Since every $\alpha$-closed (regular) open set is (regular) closed, $A$ is clopen.

Theorem 3.32. A set $A$ is $g\omega\alpha$-open in $(X, \tau)$ if and only if $F \subseteq \alpha\text{int}(A)$ whenever $F$ is $\omega\alpha$-closed in $(X, \tau)$ and $F \subseteq A$.

Proof: Suppose $F \subseteq \alpha\text{int}(A)$ where $F$ is $\omega\alpha$-closed and $F \subseteq A$. Let $X-A \subseteq G$ where $G$ is $\omega\alpha$-open in $(X, \tau)$. Then $G \subseteq X-G$ and $X-G \subseteq \alpha\text{int}(A)$. Thus $X-A$ is $g\omega\alpha$-closed in $(X, \tau)$. Hence $A$ is $g\omega\alpha$-open in $(X, \tau)$.

Conversely: Suppose that $A$ is $g\omega\alpha$-open. $F \subseteq A$ and $F$ is $\omega\alpha$-closed in $(X, \tau)$. Then $X-F$ is $\omega\alpha$-open and $X-A \subseteq X-F$. Therefore $\alpha\text{cl}(X-A) \subseteq X-F$. But $\alpha\text{cl}(X-A) = X-\alpha\text{int}(A)$. Hence $F \subseteq \alpha\text{int}(A)$.

Theorem 3.33. A subset $A$ is $g\omega\alpha$-open in $(X, \tau)$ if and only if $G = X$ whenever $G$ is $\omega\alpha$-open and $\alpha\text{int}(A) \cup (X-G) \subseteq G$.

Proof: Let $A$ be $g\omega\alpha$-open. $G$ be $\omega\alpha$-open and $\alpha\text{int}(A) \cup (X-A) \subseteq G$. This gives $X-G \subseteq (X-\alpha\text{int}(A)) \cap (X-(X-A)) = X-\alpha\text{int}(A)-(X-A) = \alpha\text{cl}(X-A)-(X-A)$. Since $X-A$ is $g\omega\alpha$-closed and $X-G$ is $\omega\alpha$-closed. Then by theorem 3.32 it follows that $X-G = \phi$. Therefore $X = G$.

Conversely: Suppose $F$ is $\omega\alpha$-closed and $F \subseteq A$. Then $\alpha\text{int}(A) \cup (X-A) \subseteq \alpha\text{int}(A) \cup (X-F)$. It follows that $\alpha\text{int}(A) \cup (X-F) = X$ and hence $F \subseteq \alpha\text{int}(A)$. Therefore $A$ is $g\omega\alpha$-open in $(X, \tau)$.
4 gωα-Closure and gωα-Interior

In this section the notion of gωα-closure and gωα-interior is defined and some of its basic properties are studied.

**Definition 4.1.** For a subset $A$ of $(X, \tau)$ $g\omega\alpha$-closure of $A$ is denoted by $g\omega\alpha cl(A)$ and is defined as $g\omega\alpha cl(A) = \bigcap \{ G; A \subseteq G, G$ is $g\omega\alpha$-closed in $(X, \tau)\}$.

**Theorem 4.2.** For an $x \in X$, $x \in g\omega\alpha cl(A)$ if and only if $A \cap V \neq \emptyset$ for every $g\omega\alpha$-open set $V$ containing $x$.

**Proof:** Let $x \in g\omega\alpha cl(A)$. Suppose there exists a $g\omega\alpha$-open set $V$ containing $x$ such that $V \cap A = \emptyset$. Then $A \subseteq X-V$, $g\omega\alpha cl(A) \subseteq X-V$. This implies $x \notin g\omega\alpha cl(A)$ which is a contradiction. Hence $A \cap V \neq \emptyset$.

Conversely, Suppose $x \notin g\omega\alpha cl(A)$ then there exists $g\omega\alpha$-closed set $G$ containing $A$ such that $x \notin G$. Then $x \in X-G$ and $X-G$ is $g\omega\alpha$-open. Also $(X-G) \cap A = \emptyset$ which is a contradiction to the hypothesis, Hence $x \in g\omega\alpha cl(A)$.

**Theorem 4.3.** If $A \subseteq X$, then $A \subseteq g\omega\alpha cl(A) \subseteq \text{cl}(A)$.

**Proof:** Since every closed set is $g\omega\alpha$-closed, the proof follows.

**Remark 4.4.** Both containment relations in the theorem 4.3 may be proper as seen from the following example.

**Example 4.5.** In Example 3.10, the set $A = \{a\}$ then $g\omega\alpha cl(A) = \{a, c\}$ and $\text{cl}(A) = X$, and so $A \subseteq g\omega\alpha cl(A) \subseteq \text{cl}(A)$.

**Theorem 4.6.** If $A$ is $g\omega\alpha$-closed, then $g\omega\alpha cl(A) = A$.

**Proof:** Let $A$ be $g\omega\alpha$-closed set in $(X, \tau)$. Since $A \subseteq A$ and $A$ is $g\omega\alpha$-closed set, $A \in \{ G; A \subseteq G, G$ is $g\omega\alpha$-closed set $\}$ which implies that $A = \bigcap \{ G; A \subseteq G, G$ is $g\omega\alpha$-closed set $\} \subseteq A$, that is $g\omega\alpha cl(A) \subseteq A$. But $A \subseteq g\omega\alpha cl(A)$ is always true. Hence $A = g\omega\alpha cl(A)$.

**Theorem 4.7.** If $A \subseteq X$ and $A$ is $g\omega\alpha$-closed, then $g\omega\alpha cl(A)$ is the smallest $g\omega\alpha$-closed subset of $X$ containing $A$.

**Proof:** Let $A$ be $g\omega\alpha$-closed set in $(X, \tau)$. Then $g\omega\alpha cl(A) = \bigcap \{ G; A \subseteq G, G$ is $g\omega\alpha$-closed in $(X, \tau)\}$. Since $A \subseteq A$ and $A$ is $g\omega\alpha$-closed set, $g\omega\alpha cl(A) = A$ is the smallest $g\omega\alpha$-closed subset of $X$ containing $A$.

However the converse of the above theorem need not be true as seen from the following example.

**Example 4.8.** In Example 3.13, the set $A = \{a, c\}$ then $g\omega\alpha cl(A) = X$, which is the smallest $g\omega\alpha$-closed set in $X$ containing $A$ but $A$ is not $g\omega\alpha$-closed in $(X, \tau)$.

**Remark 4.9.** The following example shows that for any two subsets $A$ and $B$ of $X$, $A \subseteq B$ implies $g\omega\alpha cl(A) \neq g\omega\alpha cl(B)$.
Example 4.10. In example 3.13, the set \( A = \{ c \} \) and \( B = \{ a, c \} \) then \( A \subseteq B \). Now \( g_{\omega cl}(A) = \{ c \} \) and \( g_{\omega cl}(B) = X \). Hence \( g_{\omega cl}(A) \neq g_{\omega cl}(B) \).

Remark 4.11. For a subset \( A \) of \((X, \tau)\) \( g_{\omega cl}(A) \neq \text{cl}(A) \) as seen from the following example.

Example 4.12. In Example 3.13, the set \( A = \{ c \} \subseteq X \), \( g_{\omega cl}(A) = \{ c \} \) and \( \text{cl}(A) = \{ b, c \} \). Therefore \( g_{\omega cl}(A) \neq \text{cl}(A) \).

Remark 4.13. For any two subsets \( A \) and \( B \) of \((X, \tau)\), \( g_{\omega cl}(A) = g_{\omega cl}(B) \) does not imply that \( A = B \). This is shown by the following example.

Example 4.14. In Example 3.7, the set \( A = \{ a \} \) and \( B = \{ a, c \} \) then \( g_{\omega cl}(A) = g_{\omega cl}(B) \). But \( A \neq B \).

Theorem 4.15. Let \( A \) and \( B \) be the subsets of \((X, \tau)\), Then,
1. \( g_{\omega cl}(\phi) = \phi \).
2. \( g_{\omega cl}(X) = X \).
3. \( g_{\omega cl}(A) \) is \( g_{\omega cl} \)-closed set in \((X, \tau)\).
4. If \( A \subseteq B \) then \( g_{\omega cl}(A) \subseteq g_{\omega cl}(B) \).
5. \( g_{\omega cl}(A \cup B) = g_{\omega cl}(A) \cup g_{\omega cl}(B) \).
6. \( g_{\omega cl}(g_{\omega cl}(A)) = g_{\omega cl}(A) \).

Proof: Proof of (1), (2), (3) and (4) are obvious from definition 4.1.

(5). We know that \( g_{\omega cl}(A) \subseteq g_{\omega cl}(A \cup B) \) and \( g_{\omega cl}(B) \subseteq g_{\omega cl}(A \cup B) \) \( \Rightarrow \) \( g_{\omega cl}(A) \cup g_{\omega cl}(B) \subseteq g_{\omega cl}(A \cup B) \)--(i). Now we prove \( g_{\omega cl}(A \cup B) \subseteq g_{\omega cl}(A) \cup g_{\omega cl}(B) \). Let \( x \) be any point such that \( x \notin g_{\omega cl}(A) \cup g_{\omega cl}(B) \), then there exists \( g_{\omega cl} \)-closed sets \( P \) and \( Q \) such that \( A \subseteq P \) and \( B \subseteq Q \), \( x \notin P \) and \( Q \), then \( x \notin P \cup Q \), \( A \cup B \subseteq P \cup Q \) and \( P \cup Q \) is \( g_{\omega cl} \)-closed set by Theorem 3.20, thus \( x \notin g_{\omega cl}(A \cup B) \Rightarrow g_{\omega cl}(A \cup B) \subseteq g_{\omega cl}(A) \cup g_{\omega cl}(B) \)--(ii). From (i) and (ii) \( g_{\omega cl}(A \cup B) = g_{\omega cl}(A) \cup g_{\omega cl}(B) \).

(6). Let \( P \) be \( g_{\omega cl} \)-closed set containing \( A \). Then by definition 4.1 \( g_{\omega cl}(A) \subseteq P \). Since \( P \) is \( g_{\omega cl} \)-closed set and contains \( g_{\omega cl}(A) \) and is contained in every \( g_{\omega cl} \)-closed set containing \( A \), it follows \( g_{\omega cl}(g_{\omega cl}(A)) \subseteq g_{\omega cl}(A) \). Therefore \( g_{\omega cl}(g_{\omega cl}(A)) = g_{\omega cl}(A) \).

Theorem 4.16. Let \( A \) and \( B \) be subset of \((X, \tau)\) then \( g_{\omega cl}(A \cap B) \subseteq g_{\omega cl}(A) \cap g_{\omega cl}(B) \).

Proof: Since \( A \cap B \subseteq A \) and \( A \cap B \subseteq B \), by theorem 4.15 (4), \( g_{\omega cl}(A \cap B) \subseteq g_{\omega cl}(A) \) and \( g_{\omega cl}(A \cap B) \subseteq g_{\omega cl}(B) \). Thus \( g_{\omega cl}(A \cap B) \subseteq g_{\omega cl}(A) \cap g_{\omega cl}(B) \).

In general \( g_{\omega cl}(A) \cap g_{\omega cl}(B) \subseteq g_{\omega cl}(A \cap B) \) as seen from the following example.

Example 4.17. In Example 3.18, the set \( A = \{ a \} \) and \( B = \{ b \} \) then \( g_{\omega cl}(A) = \{ a, c \} \) and \( g_{\omega cl}(B) = \{ b, c \} \) and \( g_{\omega cl}(A \cap B) = \phi \). Hence \( g_{\omega cl}(A) \cap g_{\omega cl}(B) \subseteq g_{\omega cl}(A \cap B) \).

Now we introduce the following.
Definition 4.18. For a subset $A$ of $(X, \tau)$ $g_{\omega\alpha}$-interior of $A$ is denoted by $g_{\omega\alpha}\text{int}(A)$ and is defined as $g_{\omega\alpha}\text{int}(A) = \cup \{ G; G \subseteq A \text{ and } G \text{ is } g_{\omega\alpha}\text{-open in } (X, \tau) \}$. That is $g_{\omega\alpha}\text{int}(A)$ is the union of all $g_{\omega\alpha}$-open sets contained in $A$.

Theorem 4.19. Let $A$ be subset of $(X, \tau)$ then $g_{\omega\alpha}\text{int}(A)$ is the largest $g_{\omega\alpha}$-open subset of $X$ contained in $A$ if $A$ is $g_{\omega\alpha}$-open.

Proof: Let $A \subseteq X$ be $g_{\omega\alpha}$-open, then $g_{\omega\alpha}\text{int}(A) = \cup \{ G; G \subseteq A \text{ and } G \text{ is } g_{\omega\alpha}\text{-open in } (X, \tau) \}$ Since $A \subseteq A$ and $A$ is $g_{\omega\alpha}$-open, $A = g_{\omega\alpha}\text{int}(A)$ is the largest $g_{\omega\alpha}$-open subset of $X$ contained in $A$.

The converse of the above theorem need not be true as seen from the following example.

Example 4.20. In Example 3.18, the set $A = \{ b, c \}$, then $g_{\omega\alpha}\text{int}(A) = \{ b \}$ is $g_{\omega\alpha}$-open in $(X, \tau)$, but $A$ is not $g_{\omega\alpha}$-open in $(X, \tau)$.

Remark 4.21. For any subset $A$ of $X$, $\text{int}(A) \subseteq g_{\omega\alpha}\text{int}(A) \subseteq A$.

Remark 4.22. For a subset $A$ of $X$, $g_{\omega\alpha}\text{int}(A) \neq \text{int}(A)$ as seen from the following example.

Example 4.23. In Example 3.5, the set $A = \{ b \}$, then $g_{\omega\alpha}\text{int}(A) = \{ b \}$ and $\text{int}(A) = \phi$ hence $g_{\omega\alpha}\text{int}(A) \neq \text{int}(A)$.

Remark 4.24. For any two subsets $A$ and $B$ of $X$ $g_{\omega\alpha}\text{int}(A) = g_{\omega\alpha}\text{int}(B)$ does not imply that $A = B$. That is shown by the following example.

Example 4.25. In Example 3.7, the set $A = \{ b \}$ and $B = \{ c \}$ then $g_{\omega\alpha}\text{int}(A) = \phi = g_{\omega\alpha}\text{int}(B)$. But $A \neq B$.

Remark 4.26. For any two subsets $A$ and $B$ of $X$, $g_{\omega\alpha}\text{int}(A) \cup g_{\omega\alpha}\text{int}(B) \neq g_{\omega\alpha}\text{int}(A \cup B)$.

Example 4.27. In Example 3.18 the set $A = \{ b, c \}$ and $B = \{ a, c \}$ now $g_{\omega\alpha}\text{int}(A) = \{ b \}$ and $g_{\omega\alpha}\text{int}(B) = \{ a \}$ and $g_{\omega\alpha}\text{int}(A \cup B) = g_{\omega\alpha}\text{int}X = X$. Hence $g_{\omega\alpha}\text{int}(A) \cup g_{\omega\alpha}\text{int}(B) \neq g_{\omega\alpha}\text{int}(A \cup B)$.

Theorem 4.28. For any subset $A$ of $X$ $[X-g_{\omega\alpha}\text{int}(A)] = [g_{\omega\alpha}\text{cl}(X-A)]$.

Proof: Let $X \in X-g_{\omega\alpha}\text{int}(A)$, then $X$ is not in $g_{\omega\alpha}\text{int}(A)$, that is every $g_{\omega\alpha}$-open set $G$ containing $x$ is such that $G \subseteq A$. This implies every $g_{\omega\alpha}$-open set $G$ containing $x$ intersects $X-A$. That is $G \cap (X-A) \neq \phi$. Then by theorem 4.2 $x \in g_{\omega\alpha}\text{cl}(X-A)$ and therefore $[X-g_{\omega\alpha}\text{int}(A)] \subseteq [g_{\omega\alpha}\text{cl}(X-A)]$.

Conversely; Let $x \in g_{\omega\alpha}\text{cl}(X-A)$, then every $g_{\omega\alpha}$-open set $G$ containing $x$ intersects $X-A$. That is, $G \cap (X-A) \neq \phi$. That is every $g_{\omega\alpha}$-open set $G$ containing $x$ is such that $G \subseteq A$. Then by definition 4.18, $x$ not in $g_{\omega\alpha}\text{int}(A)$, that is $x \in [X-g_{\omega\alpha}\text{int}(A)]$; and so $[g_{\omega\alpha}\text{cl}(X-A)] \subseteq [X-g_{\omega\alpha}\text{int}(A)]$. Thus $[X-g_{\omega\alpha}\text{int}(A)] = [g_{\omega\alpha}\text{cl}(X-A)]$. 
5 gωα-Neighborhoods and gωα-Limit points

In this section we define the notion of gωα-neighborhood, gωα-limit point and gωα-derived set of a set and show some of their basic properties and analogous to those for open sets.

Definition 5.1. Let (X, τ) be a topological space and let x ∈ X. A subset N of X is said to be gωα-neighborhood of a point x ∈ X if there exists an gωα-open set G such that x ∈ G ⊆ N.

Definition 5.2. Let (X, τ) be a topological space and A be a subset of X, A subset N of X is said to be gωα-neighborhood of A if there exists an gωα-open set G such that A ∈ G ⊆ N.

The collection of all gωα-neighborhood of x ∈ X is called the gωα-neighborhood system at x and shall be denoted by gωαN(x).

Theorem 5.3. A subset A of a topological space is gωα-open if it is a gωα-neighborhood of each of its points.

Proof: Let a subset G of a topological space be gωα-open. Then for every x ∈ X, x ∈ G ⊆ G, and therefore G is a gωα-neighborhood of each of its points.

The converse of the above theorem need not be true as seen from the following example.

Example 5.4. In Example 3.7 the set A = {b, c} is gωα-neighborhood of each of its points b and c but A is not gωα-open.

Theorem 5.5. Let (X, τ) be a topological space. If A is gωα-closed subset of X and x ∈ gωαcl(A) if and only if for any gωα-neighborhood N of x in (X, τ), N ∩ A ≠ φ.

Proof: Let us assume that there is a gωα-neighborhood N of the point x in (X, τ) such that N ∩ A = φ. There exist an gωα-open set G of X such that X ∈ G ⊆ N. Therefore we have G ∩ A = φ and so x ∈ X-G. Then gωαcl(A) ∈ X-G and therefore x ∉ gωαcl(A), which is the contradiction to the hypothesis x ∈ gωαcl(A). Therefore N ∩ A ≠ φ.

Conversely: Suppose that x ∉ gωαcl(A). Then there exists a gωα-closed set G of (X, τ) such that A ⊆ G and x ∉ G. Thus x ∈ X-G and X-G is gωα-open in (X, τ) and hence X-G is a gωα-neighborhood of x in (X, τ). But A ∩ (X-G) = φ which is a contradiction. Hence x ∈ gωαcl(A).

Theorem 5.6. Let (X, τ) be a topological space and x ∈ X. Let gωαN(x) be the collection of all gωα-neighborhood of x. Then,
1. gωαN(x) ≠ φ and x ∈ each member of gωαN(x).
2. The intersection of the any two members of gωαN(x) is again a member of gωαN(x).
3. If N ∈ gωαN(x) and M ⊆ N, then M ∈ gωαN(x).
4. Each member N ∈ gωαN(x) is a superset of a member G ∈ gωαN(x) where G is a gωα-open set.
Proof: (1). Since $X$ is $g\omega$-open set containing $p$, it is a $g\omega$-neighborhood of every $p \in X$. Hence there exists at least one $g\omega$-neighborhood namely $X$ for each $p \in X$ there is $g\omega N(p) \neq \emptyset$. Let $N \in g\omega N(p)$, $N$ is a $g\omega$-neighborhood of $p$, then there exists a $g\omega$-open set $G$ such that $p \in G \subseteq N$ so $p \in N$. Therefore $p \in \text{every member } N \text{ of } g\omega N(p)$.

(2). Let $N \in g\omega N(p)$ and $M \in g\omega N(p)$. Then by definition 5.1, there exists $g\omega$-open set $G$ and $F$ such that $p \in G \subseteq N$ and $p \in F \subseteq M$. Hence $p \in G \cap F \subseteq M \cap N$. Note that $G \cap F$ is a $g\omega$-neighborhood of $p$. Therefore it follows that $N \cap M$ is a $g\omega$-neighborhood of $p$. Hence $N \cap M \in g\omega N(p)$.

(3). If $N \in g\omega N(p)$ then there is an $g\omega$-open set $G$ such that $p \in G \subseteq N$. Since $M \subseteq N$, $M$ is a $g\omega$-neighborhood of $p$. Hence $M \in g\omega N(p)$.

(4). Let $N \in g\omega N(p)$ then there exists a $g\omega$-open set $G$, such that $p \in G \subseteq N$. Since $G$ is $g\omega$-open and $p \in G$, $G$ is $g\omega$-neighborhood of $P$. Therefore $G \in g\omega N(p)$ and also $G \subseteq N$.

Definition 5.7. Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$. Then a point $x \in X$ is called a $g\omega$-limit point of $A$ if and only if every $g\omega$-neighborhood of $x$ contains a point of $A$ distinct from $x$. That is $[N-\{x\}] \cap A \neq \emptyset$ for each $g\omega$-neighborhood $N$ of $x$. Also equivalently if and only if every $g\omega$-open set $G$ containing $x$ contains a point of $A$ other then $x$.

In a topological space $(X, \tau)$ the set of all $g\omega$-limit points of a given subset $A$ of $X$ is called a $g\omega$-derived set of $A$ and is denoted by $g\omega d(A)$.

Theorem 5.8. Let $A$ and $B$ be subset of a topological space $(X, \tau)$. Then,

1. $g\omega d(\emptyset) = \emptyset$.
2. If $A \subseteq B$, then $g\omega d(A) \subseteq g\omega d(B)$.
3. If $x \in g\omega d(A)$, then $x \in g\omega d(A-\{x\})$.
4. $g\omega d(A \cup B) = g\omega d(A) \cup g\omega d(B)$.
5. $g\omega d(A \cap B) \subseteq g\omega d(A) \cap g\omega d(B)$.

Proof: (1). Let $x$ be any point of $X$ and $x \in g\omega d(\emptyset)$. That is $x$ is a $g\omega$-limit point of $\emptyset$. Then for every $g\omega$-open set $G$ containing $x$, we should have $[G-\{x\}] \cap \emptyset \neq \emptyset$ which is impossible. Hence $g\omega d(\emptyset) = \emptyset$.

(2). If $x \in g\omega d(A)$, that is if $x$ is a $g\omega$-limit point of $A$, then by Definition 5.7 $[G-\{x\}] \cap A \neq \emptyset$ for every $g\omega$-open set $G$ containing $x$. Since $A \subseteq B$ implies $[G-\{x\}] \cap A \subseteq [G-\{x\}] \cap B$. Thus if $x$ is a $g\omega$-limit point of $A$ it is also a $g\omega$-limit point of $B$, that is $x \in g\omega d(B)$. Hence $g\omega d(A) \subseteq g\omega d(B)$.

(3). If $x \in g\omega d(A)$, by definition 5.7 every $g\omega$-open set $G$ containing $x$ contains at least one point other than $x$ of $A-\{x\}$. Hence $x$ is a $g\omega$-limit point of $A-\{x\}$ and it belongs to $g\omega d[A-\{x\}]$. Therefore $x \in g\omega d(A) \Rightarrow x \in g\omega d[A-\{x\}]$.

(4). Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$, from (1) $g\omega d(A) \cup g\omega d(B) \subseteq g\omega d(A \cup B)$. To prove other way if $x \notin g\omega d(A) \cup g\omega d(B)$, then $x \notin g\omega d(A)$ and $x \notin g\omega d(B)$. Hence there exists $g\omega$-neighborhoods $G_1$ and $G_2$ of $x$ such that $G_1 \cap [A-\{x\}] = \emptyset$ and $G_2 \cap (B-\{x\}) = \emptyset$. Since $G_1 \cap G_2$ is $g\omega$-neighborhood of $x$, we have $(G_1 \cap G_2) \cap [(A \cup B)-\{x\}] = \emptyset$. Therefore $x \notin g\omega d(A \cup B)$. Hence $g\omega d(A \cup B) = g\omega d(A) \cup g\omega d(B)$.

(5). Since $A \cup B \subseteq A$ and $A \cap B \subseteq B$, by (2) $g\omega d(A \cap B) \subseteq g\omega d(A)$ and $g\omega d(A \cap B) \subseteq g\omega d(B)$. Consequently $g\omega d(A \cap B) \subseteq g\omega d(A) \cap g\omega d(B)$.
Theorem 5.9. Let $(X, \tau)$ be a topological space and $A$ be a subset of $X$. If $A$ is $g\omega\alpha$-closed, then $g\omega\alpha d(A) \subseteq A$.

Proof: Let $A$ be $g\omega\alpha$-closed. Now we will show that $g\omega\alpha d(A) \subseteq A$. Since $A$ is $g\omega\alpha$-closed, $X-A$ is $g\omega\alpha$-open. To each $x \in X-A$ there exists $g\omega\alpha$-neighborhood $G$ of $x$ such that $G \subseteq X-A$. Since $A \cap (X-A) = \phi$, the $g\omega\alpha$-neighborhood $G$ contains no point of $A$ and so $X$ is not a $g\omega\alpha$-limit point of $A$. Thus no point of $X-A$ can be $g\omega\alpha$-limit point of $A$ that is, $A$ contains all its $g\omega\alpha$-limit points. that is $g\omega\alpha d(A) \subseteq A$.

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