Some Properties of Contra $gb$-continuous Functions

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Abstract

We introduce some properties of functions called contra $gb$-continuous function which is a generalization of contra $b$-continuous functions [3]. Some characterizations and several properties concerning contra $gb$- continuous functions are obtained.

Keywords: $g$-open, $g$-continuity, contra $gb$-continuity.

1 Introduction

In 1996, Dontchev [16] introduced the notion of contra continuous functions. In 2007, Caldas, Jafari, Noiri and Simoes [10] introduced a new class of functions called generalized contra continuous (contra $g$-continuous) functions. They defined a function $f : X \rightarrow Y$ to be contra $g$- continuous if preimage every open subset of $Y$ is $g$-closed in $X$. New types of contra generalized continuity such as contra $ag$-continuity [23] and contra $gs$-continuity [17] have been introduced and investigated. Recently, Nasef [30] introduced and studied so-called contra $b$-continuous functions. After that in 2009, Omari and

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Noorani [4] have studied further properties of contra $b$-continuous functions. The purpose of the present paper is to introduce some properties of notion of contra generalized $b$–continuity (contra $gb$–continuity) via the concept of $gb$–open sets in [3] and investigate some of the fundamental properties of contra $gb$–continuous functions. It turns out that contra $gb$–continuity is stronger than contra $g\beta$–continuity and weaker than both contra $gp$–continuity and contra $gs$–continuity [17].

2 Preliminaries

Throughout the paper, the space $X$ and $Y$ (or $(X, \tau)$ and $(Y, \sigma)$) stand for topological spaces with no separation axioms assumed unless otherwise stated. Let $A$ be a subset of a space $X$. The closure and interior of $A$ are denoted by $cl(A)$ and $int(A)$, respectively.

**Definition 2.1.** A subset $A$ of a space $X$ is said to be:

(a) regular open [33] if $A = int(cl(A))$
(b) $\alpha$–open [31] if $A \subset int(cl(int(A)))$
(c) semi-open [24] if $A \subset cl(int(A))$
(d) pre-open [28] or nearly open [19] if $A \subset int(cl(A))$
(e) $\beta$–open [1] or semi-$\beta$–open [6] if $A \subset cl(int(cl(A)))$
(f) $b$–open [7] or $sp$–open [18] or $\gamma$–open [19] if $A \subset cl(int(A)) \cup int(cl(A))$.

The family of all semi-open (resp. pre-open, $\alpha$-open, $\beta$–open, $\gamma$–open) sets of $(X, \tau)$ will be denoted by $SO(X, \tau)$ (resp. $PO(X, \tau)$, $\alpha O(X, \tau)$, $\beta O(X, \tau)$, $\gamma O(X, \tau)$). It is shown in [31] that $\alpha O(X, \tau)$ is a topology denoted by $\tau^\alpha$ and it is stronger than the given topology on $X$. The complement of a regular-open (resp. semi-open, pre-open, $\alpha$-open, $\beta$-open, $\gamma$-open) set is said to be regular closed (resp. semi-closed, preclosed, $\alpha$-closed, $\beta$-closed, $\gamma$-closed). The collection of all closed subsets of $X$ will be denoted by $C(X)$. We set $C(X, x) = \{ V \in C(X) : x \in V \}$ for $x \in X$. We define similarly $\gamma O(X, x)$.

The complement of $b$-open set is said to be $b$-closed [7]. The intersections of all $b$-closed sets of $X$ containing $A$ is called the $b$–closure of $A$ and is denoted by $bel(A)$. The union of all $b$-closed sets $X$ contained in $A$ is called $b$-interior of $A$ and is denoted by $bint(A)$.

**Definition 2.2.** [30] A function $f : (X, \tau) \to (Y, \sigma)$ is called contra $b$-continuous if the preimage of every open subset of $Y$ is $b$-closed in $X$.

**Definition 2.3.** [21] Let $X$ be a space. A subset $A$ of $X$ is called a generalized $b$-closed set (simply; $gb$-closed set) if $bel(A) \subset U$ whenever $A \subset U$ and $U$ is open.

The complement of a generalized $b$-closed set is called generalized $b$-open (simply; $gb$-open). Every $b$-closed set is $gb$-closed, but the converse is not true. And the collection of all $gb$-closed (resp. $gb$-open) subsets of $X$ is denoted by $gbC(X)$ (resp. $gbO(X)$).

**Example 2.4.** [5] Let $X = \{a, b, c\}$ and let $\tau = \{\emptyset, \{a\}, X\}$, then the family of all $b$-closed set of $X$ is $bC(X) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{b, a\}, \{a, c\}\}$ but the family of all $gb$-closed set of $X$ is $gbC(X) = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, c\}\}$. Then it is clear that $\{a, c\}$ is $gb$-closed but not $b$-closed in $X$.  

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Lemma 2.5. Let \((X, \tau)\) be a topological space.
(a) The intersections of a \(b\)-open set and a \(gb\)-open set is a \(gb\)-open set.
(b) The union of any family of \(gb\)-open sets is a \(gb\)-open set.

**Proof.** The statements are proved by using the same method as in proving the corresponding results for the class of \(b\)-open sets (see [7]).

### 3 Contra \(gb\)-continuous functions

In this section, we introduce some properties of continuity called contra \(gb\)-continuity which is weaker than both of contra \(gs\)-continuity and contra \(gp\)-continuity and stronger than contra \(g\beta\)-continuity.

**Definition 3.1.** [3] A function \(f : (X, \tau) \to (Y, \sigma)\) is called contra \(gb\)-continuous if the preimage of every open subset of \(Y\) is \(gb\)-closed in \(X\).

**Corollary 3.2.** If a function \(f : (X, \tau) \to (Y, \sigma)\) is contra \(b\)-continuous, then \(f\) is contra \(gb\)-continuous.

**Proof.** Obvious.

Note that the converse of the above is not necessary true as shows by the following example:

**Example 3.3.** Let \(X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}\) and \(\sigma = \{\emptyset, \{a, c\}, X\}\). Then the identity function \(f : (X, \tau) \to (X, \sigma)\) is contra \(gb\)-continuous but not contra \(b\)-continuous, since \(A = \{a, c\} \in \sigma\) but \(A\) is not \(b\)-closed in \((X, \tau)\).

**Definition 3.4.** Let \(A\) be a subset of a space \((X, \tau)\).
(a) The set \(\cap\{U \in \tau : A \subset U\}\) is called the kernel of \(A\) [29] and is denoted by \(\ker(A)\). In [25] the kernel of \(A\) is called the \(\Lambda\)-set.
(b) The set \(\cap\{F \subset X : A \subset F, F\) is \(gb\)-closed\} is called the \(gb\)-closure of \(A\) and is denoted by \(\text{gbcl}(A)\) [21].
(c) The set \(\cup\{G \subset X : G \subset A, G\) is \(gb\)-open\} is called the \(gb\)-interior of \(A\) and is denoted by \(\text{gbint}(A)\) [21].

**Lemma 3.5.** For an \(x \in X, x \in \text{gbcl}(A)\) if and only if \(U \cap A \neq \emptyset\) for every \(gb\)-open set \(U\) containing \(x\).

**Proof.** (Necessity) Suppose there exists a \(gb\)-open set \(U\) containing \(x\) such that \(U \cap A = \emptyset\). Since \(A \subset X - U, \text{gbcl}(A) \subset X - U\). This implies \(x \notin \text{gbcl}(A)\), a contradiction.
(Sufficiency) Suppose \(x \notin \text{gbcl}(A)\). Then there exists a \(gb\)-closed subset \(F\) containing \(A\) such that \(x \notin F\). Then \(x \in X - F\) and \(X - F\) is \(gb\)-open also \((X - F) \cap A = \emptyset\), a contradiction.
Lemma 3.6. [22] The following properties hold for subsets $A, B$ of a space $X$:

(a) $x \in \ker(A)$ if and only if $A \cap F \neq \emptyset$ for any $F \in C(X, x)$.
(b) $A \subset \ker(A)$ and $A = \ker(A)$ if $A$ is open in $X$.
(c) If $A \subset B$, then $\ker(A) \subset \ker(B)$.

Theorem 3.7. For a function $f : (X, \tau) \to (Y, \sigma)$, the following continuities are equivalent:

(a) $f$ is contra $gb$-continuous;
(b) For every closed subsets $F$ of $Y$, $f^{-1}(F) \in gbO(X, x)$;
(c) For each $x \in X$ and each $F \in C(Y, f(x))$, there exists $U \in gbO(X, x)$ such that $f(U) \subset F$;
(d) $f(gbcl(A)) \subset \ker(f(A))$ for every subset $A$ of $X$;
(e) $gbcl(f^{-1}(B)) \subset f^{-1}(\ker(B))$ for every subset $B$ of $Y$.

Proof. The implications $(a) \Leftrightarrow (b)$ and $(b) \Rightarrow (c)$ are obvious.

$(c) \Rightarrow (b)$: Let $F$ be any closed set of $Y$ and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in gbO(X, x)$ such that $f(U_x) \subset F$. Therefore, we obtain $f^{-1}(F) = \cup \{U_x : x \in f^{-1}(F)\}$ which is $gb$-open in $X$.

$(b) \Rightarrow (d)$: Let $A$ be any subset of $X$. Suppose that $y \notin \ker(f(A))$. Then by Lemma 3.6 there exists $F \in C(Y, y)$ such that $f(A) \cap F = \emptyset$. Thus, we have $A \cap f^{-1}(F) = \emptyset$ and $gbcl(A) \cap f^{-1}(F) = \emptyset$. Therefore, we obtain $f(gbcl(A)) \cap F = \emptyset$ and $y \notin f(gbcl(A))$. This implies that $f(gbcl(A)) \subset \ker(f(A))$.

$(d) \Rightarrow (e)$: Let $B$ be any subset of $Y$. By $(d)$ and Lemma 3.6, we have $f(gbcl(f^{-1}(B))) \subset \ker(f^{-1}(B)) \subset \ker(B)$ and $gbcl(f^{-1}(B)) \subset f^{-1}(\ker(B))$.

$(e) \Rightarrow (a)$: Let $V$ be any open set of $Y$. Then, by Lemma 3.6, we have $gbcl(f^{-1}(V)) \subset f^{-1}(V)$ and $gbcl(f^{-1}(V)) = f^{-1}(V)$. This shows that $f^{-1}(V)$ is $gb$-closed in $X$. \qed

Definition 3.8. [4] A function $f : (X, \tau) \to (Y, \sigma)$ is called $gb$-continuous if the preimage of every open subset of $Y$ is $gb$-open in $X$.

Remark 3.9. The following two examples will show that the concept of $gb$-continuity and contra $gb$-continuity are independent from each other.

Example 3.10. Let $X = \{a, b\}$ be the Sierpinski space with the topology $\tau = \{\emptyset, \{a\}, X\}$. Let $f : (X, \tau) \to (X, \tau)$ be defined by: $f(a) = b$ and $f(b) = a$. It can be easily observed that $f$ is contra $gb$-continuous. But $f$ is not $gb$-continuous, since $\{a\}$ is open and its preimage $\{b\}$ is not $gb$-open.

Example 3.11. The identity function on the real line with the usual topology is continuous [23, Example 2] and hence $gb$-continuous. The inverse image of $(0, 1)$ is not $gb$-closed and the function is not contra $gb$-continuous.

Definition 3.12. A subset $A$ of a space $(X, \tau)$ is called

(a) a generalized semiclosed set (briefly gs-closed) [8] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open;
(b) an $\alpha$-generalized closed set (briefly $\alpha$g-closed) [25] if $acl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open;
(c) a generalized pre-closed set (briefly gp-closed) [26] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open;
(d) a generalized $\beta$-closed set (briefly $g\beta$-closed) [12] if $\beta cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

Definition 3.13. A function $f : (X, \tau) \to (Y, \sigma)$ is called contra $\alpha q$-continuous [23] (resp. contra $gs$-continuous [17], contra $gp$-continuous, contra $g\beta$-continuous) if the preimage of every open subset of $Y$ is $\alpha q$-closed (resp. $gs$-closed, $gp$-closed, $g\beta$-closed) in $X$.

We obtain the following diagram by using Definition 2.1, 2.3, 3.1, 3.12 and 3.13.

\[
\begin{array}{ccc}
\text{contra continuous} & \downarrow & \text{contra $\alpha q$-continuous} \\
\text{contra $gs$-continuous} & \leftarrow & \text{contra $gp$-continuous} \\
\text{contra $gb$-continuous} & \leftarrow & \text{contra $g\beta$-continuous}
\end{array}
\]

However, the converses are not true in general as shown by the following examples.

Example 3.14. Let $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, X\}$ and $\sigma = \{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is contra $\alpha q$-continuous but not contra continuous.

Example 3.15. Let $X = \{a, b\}$ with the indiscrete topology $\tau$ and $\sigma = \{\emptyset, \{a\}, X\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is contra $gb$-continuous but not contra $gs$-continuous, since $A = \{a\} \in \sigma$ but $A$ is not $gs$-closed in $(X, \tau)$.

Example 3.16. Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, b, c\}, \{b, c, d\}, X\}$. Define a function $f : (X, \tau) \to (X, \tau)$ as follows: $f(a) = b$, $f(b) = a$, $f(c) = d$ and $f(d) = c$. Then $f$ is contra $gs$-continuous. However, $f$ is not contra $\alpha q$-continuous, since $\{c, d\}$ is a closed set of $(X, \tau)$ and $f^{-1}(\{c, d\}) = \{c, d\}$ is not $\alpha q$-open.

Example 3.17. Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $Y = \{1, 2\}$ be the Sierpinski space with the topology $\sigma = \{\emptyset, \{1\}, Y\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be defined by: $f(a) = 1$ and $f(b) = f(c) = 2$. Then $f$ is contra $gb$-continuous but not contra $gp$-continuous.

Theorem 3.18. If a function $f : X \to Y$ is contra $gb$-continuous and $Y$ is regular, then $f$ is $gb$-continuous.

Proof. Let $x$ be an arbitrary point of $X$ and $V$ be an open set of $Y$ containing $f(x)$. Since $Y$ is regular, there exists an open set $G$ in $Y$ containing $f(x)$ such that $cl(G) \subseteq V$. Since $f$ is contra $gb$-continuous, so by Theorem 3.7 there exists $U \in gbO(X, x)$ such that $f(U) \subseteq cl(G)$. Then $f(U) \subseteq cl(G) \subseteq V$. Hence, $f$ is $gb$-continuous. 

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**Definition 3.19.** A space $(X, \tau)$ is said to be:

(a) gb-space if every gb-open set of $X$ is open in $X$,

(b) locally gb-indiscrete if every gb-open set of $X$ is closed in $X$.

The following two results follow immediately from Definition 3.19.

**Theorem 3.20.** If a function $f : X \to Y$ is contra gb-continuous and $X$ is gb-space, then $f$ is contra continuous.

**Proof.** Let $V \in O(Y)$. Then $f^{-1}(V)$ is gb-closed in $X$. Since $X$ is gb-space, $f^{-1}(V)$ is closed in $X$. Thus, $f$ is contra continuous. \[\square\]

**Theorem 3.21.** Let $X$ be locally gb-indiscrete. If a function $f : X \to Y$ is contra gb-continuous, then it is continuous.

**Proof.** Let $V \in O(Y)$. Then $f^{-1}(V)$ is gb-closed in $X$. Since $X$ is locally gb-indiscrete space, $f^{-1}(V)$ is open in $X$. Thus, $f$ is continuous. \[\square\]

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subset X \times Y$ is called the graph of $f$ and is denoted by $G_f$.

**Definition 3.22.** The graph $G_f$ of a function $f : X \to Y$ is said to be contra gb-closed if for each $(x, y) \in (X \times Y) - G_f$ there exists $U \in gbO(X, x)$ and $V \in C(Y, y)$ such that $(U \times V) \cap G_f = \emptyset$.

**Lemma 3.23.** The graph $G_f$ of a function $f : X \to Y$ is contra gb-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G_f$ there exist $U \in gbO(X, x)$ and $V \in C(Y, y)$ such that $f(U) \cap V = \emptyset$.

**Theorem 3.24.** If a function $f : X \to Y$ is contra gb-continuous and $Y$ is Urysohn, then $G_f$ is contra gb-closed in the product space $X \times Y$.

**Proof.** Let $(x, y) \in (X \times Y) - G_f$. Then $y \neq f(x)$ and there exist open sets $H_1, H_2$ such that $f(x) \in H_1$, $y \in H_2$ and $cl(H_1) \cap cl(H_2) = \emptyset$. From hypothesis, there exists $V \in gbO(X, x)$ such that $f(V) \subset cl(H_1)$. Therefore, we obtain $f(V) \cap cl(H_2) = \emptyset$. This shows that $G_f$ is contra gb-closed. \[\square\]

**Theorem 3.25.** If $f : X \to Y$ is gb-continuous and $Y$ is $T_1$, then $G_f$ is contra gb-closed in $X \times Y$.

**Proof.** Let $(x, y) \in (X \times Y) - G_f$. Then $y \neq f(x)$ and there exist open set $V$ of $Y$ such that $f(x) \in V$ and $y \notin V$. Since $f$ is gb-continuous, there exists $U \in gbO(X, x)$ such that $f(U) \subseteq V$. Therefore, we obtain $f(U) \cap (Y - V) = \emptyset$ and $(Y - V) \in C(Y, y)$. This shows that $G_f$ is contra gb-closed in $X \times Y$. \[\square\]

**Definition 3.26.** [16] A space $X$ is said to be strongly $S$-closed if every closed cover of $X$ has a finite subcover.
Theorem 3.27. If \((X, \tau_{gb})\) is a topological space and \(f : X \to Y\) has a contra gb-closed graph, then the inverse image of a strongly S-closed set \(A\) of \(Y\) is gb-closed in \(X\).

Proof. Assume that \(A\) is a strongly S-closed set of \(Y\) and \(x \notin f^{-1}(A)\). For each \(a \in A\), \((x, a) \notin G_f\). By Lemma 3.23 there exist \(U_a \in \text{gbO}(X, x)\) and \(V_a \in C(Y, a)\) such that \(f(U_a) \cap V_a = \emptyset\). Then \((A \cap V_a : a \in A)\) is a closed cover of the subspace \(A\), since \(A\) is strongly S-closed, then there exists a finite subset \(A_0 \subset A\) such that \(A \subset \bigcup\{V_a : a \in A_0\}\). Set \(U = \cap\{U_a : a \in A_0\}\), but \((X, \tau_{gb})\) is a topological space, then \(U \in \text{gbO}(X, x)\) and \(f(U) \cap A \subset f(U_a) \cap \bigcup\{V_a : a \in A_0\}\) = \(\emptyset\). Therefore, \(U \cap f^{-1}(A) = \emptyset\) and hence \(x \notin \text{gbcl}(f^{-1}(A))\). This show that \(f^{-1}(A)\) is gb-closed.

Theorem 3.28. Let \(Y\) be a strongly S-closed space. If \((X, \tau_{gb})\) is a topological space and \(f : X \to Y\) has a contra gb-closed graph, then \(f\) is contra gb-continuous.

Proof. Suppose that \(Y\) is strongly S-closed and \(G_f\) is contra gb-closed. First we show that an open set of \(Y\) is strongly S-closed. Let \(U\) be an open set of \(Y\) and \(\{V_i : i \in I\}\) be a cover of \(U\) by closed sets \(V_i\) of \(U\). For each \(i \in I\), there exists a closed set \(K_i\) of \(X\) such that \(V_i = K_i \cap U\). Then the family \(\{K_i : i \in I\} \cup (Y - U)\) is a closed cover of \(Y\). Since \(Y\) is strongly S-closed, there exists a finite subset \(I_0 \subset I\) such that \(Y = \cup\{K_i : i \in I_0\} \cup (Y - U)\). Therefore, we obtain \(U = \cup\{V_i : i \in I_0\}\). This shows that \(U\) is strongly S-closed. By Theorem 3.27, \(f^{-1}(U)\) is gb-closed in \(X\) for every open \(U\) in \(Y\). Therefore, \(f\) is contra gb-continuous.

Theorem 3.29. Let \(f : X \to Y\) be a function and \(g : X \to X \times Y\) the graph function of \(f\), defined by \(g(x) = (x, f(x))\) for every \(x \in X\). If \(g\) is contra gb-continuous, then \(f\) is contra gb-continuous.

Proof. Let \(U\) be an open set in \(Y\), then \(X \times U\) is an open set in \(X \times Y\). Since \(g\) is contra gb-continuous. It follows that \(f^{-1}(U) = g^{-1}(X \times U)\) is an gb-closed in \(X\). Thus, \(f\) is contra gb-continuous.

Theorem 3.30. \(f : X \to Y\) is contra gb-continuous, \(g : X \to Y\) contra continuous, and \(Y\) is Urysohn, then \(E = \{x \in X : f(x) = g(x)\}\) is gb-closed in \(X\).

Proof. Let \(x \in X - E\). Then \(f(x) \neq g(x)\). Since \(Y\) is Urysohn, there exists open sets \(V\) and \(W\) such that \(f(x) \in V, g(x) \in W\) and \(cl(V) \cap cl(W) = \emptyset\). Since \(f\) is contra gb-continuous, then \(f^{-1}(cl(V))\) is gb-open in \(X\) and \(g\) is contra continuous, then \(g^{-1}(cl(W))\) is open in \(X\). Let \(U = f^{-1}(cl(V))\) and \(G = g^{-1}(cl(W))\). Then \(U\) and \(G\) contain \(x\). Set \(A = U \cap G\) is gb-open in \(X\). And \(f(A) \cap g(A) \subset f(U) \cap g(G) \subset cl(V) \cap cl(W) = \emptyset\). Hence \(f(A) \cap g(A) = \emptyset\) and \(A \cap E = \emptyset\) where \(A\) is gb-open therefore \(x \notin \text{gbcl}(E)\). Thus \(E\) is gb-closed in \(X\).

Theorem 3.31. Let \(\{X_i : i \in I\}\) be any family of topological spaces. If \(f : X \to \Pi X_i\) is a contra gb-continuous function. Then \(P_i \circ f : X \to X_i\) is contra gb-continuous for each \(i \in I\), where \(P_i\) is the projection of \(\Pi X_i\) onto \(X_i\).

Proof. We shall consider a fixed \(i \in I\). Suppose \(U_i\) is an arbitrary open set in \(X_i\). Then \(P_i^{-1}(U_i)\) is open in \(\Pi X_i\). Since \(f\) is contra gb-continuous, \(f^{-1}(P_i^{-1}(U_i)) = (P_i \circ f)^{-1}(U_i)\) is gb-closed in \(X\). Therefore \(P_i \circ f\) is contra gb-continuous.
Theorem 3.32. If \( f : X \to Y \) is a contra gb-continuous function and \( g : Y \to Z \) is a continuous function, then \( g\circ f : X \to Z \) is contra gb-continuous.

Proof. Let \( V \in O(Y) \). Then \( g^{-1}(V) \) is open in \( Y \). Since \( f \) is contra gb-continuous, 
\[ f^{-1}(g^{-1}(V)) = (g\circ f)^{-1}(V) \] is gb-closed in \( X \). Therefore, \( g\circ f : X \to Z \) is contra gb-continuous.

Definition 3.33. A function \( f : X \to Y \) is said to be:
(a) [21] gb-irresolute if the preimage of a gb-open subset of \( Y \) is a gb-open subset of \( X \),
(b) pre–gb-open if image of every gb-open subset of \( X \) is gb-open.

Theorem 3.34. Let \( f : X \to Y \) be surjective gb-irresolute and pre–gb-open and \( g : Y \to Z \) be any function. Then \( g\circ f : X \to Z \) is contra gb-continuous if and only if \( g \) is contra gb-continuous.

Proof. The “if” part is easy to prove. To prove the “only if” part, let \( g\circ f : X \to Z \) be contra gb-continuous and let \( F \) be a closed subset of \( Z \). Then \( (g\circ f)^{-1}(F) \) is a gb-open subset of \( X \). That is \( f^{-1}(g^{-1}(F)) \) is gb-open. Since \( f \) is pre–gb-open \( f(f^{-1}(g^{-1}(F))) \) is a gb-open subset of \( Y \). So, \( g^{-1}(F) \) is gb-open in \( Y \). Hence \( g \) is contra gb-continuous.

4 Applications

Definition 4.1. A topological space \( X \) is said to be:
(a) gb-normal if each pair of non-empty disjoint closed sets can be separated by disjoint gb-open sets,
(b) ultranormal [32] if each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets.

Theorem 4.2. If \( f : X \to Y \) is a contra gb-continuous, closed injection and \( Y \) is ultranormal, then \( X \) is gb-normal.

Proof. Let \( F_1 \) and \( F_2 \) be disjoint closed subsets of \( X \). Since \( f \) is closed and injective, 
\( f(F_1) \) and \( f(F_2) \) are disjoint closed subsets of \( Y \). Since \( Y \) is ultranormal \( f(F_1) \) and \( f(F_2) \) are separated by disjoint clopen sets \( V_1 \) and \( V_2 \), respectively. Hence \( F_1 \subset f^{-1}(V_1), F_2 \subset f^{-1}(V_2) \in gbO(X) \) and \( f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset \). Thus \( X \) is gb-normal.

Definition 4.3. [9] A topological space \( X \) is said to be gb-connected if \( X \) is not the union of two disjoint non-empty gb-open subsets of \( X \).

Theorem 4.4. A contra gb-continuous image of a gb-connected space is connected.

Proof. Let \( f : X \to Y \) be a contra gb-continuous function of a gb-connected space \( X \) onto a topological space \( Y \). If possible, let \( Y \) be disconnected. Let \( A \) and \( B \) form a disconnectedness of \( Y \). Then \( A \) and \( B \) are clopen and \( Y = A \cup B \) where \( A \cap B = \emptyset \). Since \( f \) is contra gb-continuous, \( X = f^{-1}(Y) = f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B) \) where \( f^{-1}(A) \) and \( f^{-1}(B) \) are non-empty gb-open sets in \( X \). Also, \( f^{-1}(A) \cap f^{-1}(B) = \emptyset \). Hence \( X \) is non-gb-connected which is a contradiction. Therefore \( Y \) is connected.
**Theorem 4.5.** Let $X$ be gb-connected and $Y$ be $T_1$. $f : X \to Y$ is a contra gb-continuous, then $f$ is constant.

**Proof.** Since $Y$ is $T_1$ space, $v = \{f^{-1}(y) : y \in Y\}$ is disjoint gb-open partition of $X$. If $|v| \geq 2$, then $X$ is the union of two non-empty gb-open sets. Since $X$ is gb-connected, $|v| = 1$. Therefore, $f$ is constant. 

**References**


